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NUMERICAL METHOD AND FORTRAN PROGRAM FOR SOLVING POISSON'S EQUATION OVER AXISYMMETRIC REGIONS OF A SPHERE

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16. Abstract

This report describes a numerical method and FORTRAN program for solving Poisson's equation with a specified source term, in axisymmetric regions of a sphere bounded by the spherical coordinates $r \leq 1$, $0 \leq \theta_0 < \theta \leq \pi$. Irregular mesh spacing is allowed in both the r and θ directions, and the associated matrix equation is solved by a direct (noniterative) method. For a mesh with a maximum allowable number of 376 points, the program returns a solution in 6 to 7 seconds (IBM DCS 7094/7044 computer) and requires approximately 10 000 storages.

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NUMERICAL METHOD AND FORTRAN PROGRAM FOR SOLVING POISSON'S EQUATION OVER AXISYMMETRIC REGIONS OF A SPHERE

by Oliver W. Reese

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SUMMARY

This report describes a numerical method and FORTRAN program for solving Poisson's equation in axisymmetric regions of a sphere. The associated matrix equation is solved by a direct (noniterative) method. Two geometries are considered:

(1) A full sphere with boundary values specified on the surface only.

(2) A spherical cone with additional boundary values specified on the conical surface and, optionally, along the interior negative polar axis.

In developing the numerical method, a curvilinear cell is used at each mesh point to derive the finite difference equation. The points are then ordered so as to produce a block tridiagonal symmetric matrix when the difference equation is written at all points. Finally, the particular properties of this matrix are used to write an algorithm from which the computer program is produced.

The program will accept various combinations of boundary conditions and spherical shapes. Irregular mesh spacing is permitted in both the r and θ directions. The program is very easy to use and general enough to be used as a "black box" where the solution to Poisson's equation is needed in spherical regions. The numerical method is also general enough for easy extension to other types of equations and other geometries.

INTRODUCTION

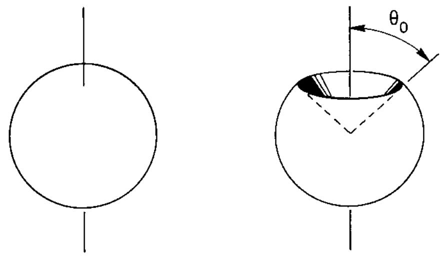
In designing a multielectrode spherical collector for a microwave amplifier tube, part of the problem required the solution to Poisson's equation for the electric potential distribution in the collector. This report describes the numerical method and the associated FORTRAN computer program for solving this equation in axisymmetric regions of a sphere. Two geometries are considered:

(1) A full sphere with boundary values prescribed on the surface only.

(2) A spherical cone, where additional boundary values are prescribed on the conical surface and, optionally, along the interior negative polar axis.

These two shapes are illustrated in figure 1.

The following approach was used in developing the numerical method. First, a curvilinear cell was used at each mesh point to derive a finite difference analog to Poisson's equation. Next, the mesh points were ordered in such a manner as to produce a block tridiagonal symmetric matrix when the finite difference equation was written at all points in the region. Finally, the particular properties of this type of matrix were used advantageously to produce a fast, efficient, and compact computer program. A direct rather than iterative method is particularly desirable in this problem since the program is used iteratively as a subroutine by a much larger calling program. This calling program continually calculates and feeds the source term used by the subroutine.



(a) Full sphere. (b) Spherical cone, $\theta_0 < \theta < \pi$.

Figure 1. - Spherical geometries considered.

The program is flexible in two ways:

(1) It will accept various combinations of boundary conditions and spherical shapes. This feature was needed to determine the optimum design for an efficient collector.

(2) Irregular mesh spacing is permitted in both the r and θ directions. This flexibility is necessary since the solution of the problem described was nonuniform in character. There were regions where it varied slowly and regions where it varied rapidly. The area of particular interest in this study was immediately adjacent to and along the negative polar axis where the electron beam was entering the collector. Here, a relatively small mesh spacing was required to obtain the solution in sufficient detail. Since the use of such a small mesh throughout the entire region would be wasteful, both of computer storage and execution time, an irregular mesh of increasing coarseness in both directions was used in moving away from this axial region. Such gradation was

also desirable in maintaining a reasonably smooth transition from mesh point to mesh point in the solution.

In addition to the features just described, the program is easy to use and general enough to recommend it for use as a black box in any situation where the solution to Poisson's equation is needed in spherical regions.

The numerical method is also general enough for easy extension to other geometries and equations where a fast, compact program is needed.

The report first explains the numerical method by discussing the derivation of the finite difference equation, the mesh point ordering, the generation of the tridiagonal block matrix, and the development of the algorithm used.

Finally, a general description of the FORTRAN computer program is given together with instructions on its use. The appendixes show such details as listing of the FORTRAN program, the results of sample problem execution, the flow charts, the FORTRAN symbol list.

SYMBOLS

(All quantities are dimensionless unless otherwise specified.)

A_i, B_i, C_i	tridiagonal elements of M
dS	element of area
dV	element of volume
F	coefficient of source term in Poisson's equation
f	source term in Poisson's equation
G_i, K_i	column vectors
L, R, T, B, D	left, right, top, bottom, and central coefficients of the unknown at any general mesh point
M	block diagonal matrix
$\vec{n}$	outward unit normal
r	radius vector in spherical coordinates
$\Delta r, \Delta\theta, \Delta\phi$	incremental change in spherical coordinates
$\bar{r}$	average value of r
r_1, r_4	radius vector at midpoint of faces 1 and 4 of curvilinear cube
$\vec{r}_1$	unit radius vector in spherical coordinates
S	total surface of curvilinear cube

S_r, S_θ, S_ϕ	faces of curvilinear cube pierced by r, θ, and ϕ axes, respectively
$\Delta S_r, \Delta S_\theta, \Delta S_\phi$	areas of faces of curvilinear cube pierced by the r, θ, and ϕ axes, respectively
$\Delta S_{r_1}, \Delta S_{r_4}$	areas of faces 1 and 4 of curvilinear cube
$\Delta S_{\theta_2}, \Delta S_{\theta_5}$	areas of faces 2 and 5 of curvilinear cube
u	electric potential, V
u_r, u_θ, u_ϕ	partial derivatives of u with respect to r, θ, and ϕ
u_{rr}	$\partial^2 u/\partial r^2$
$u_{\theta\theta}$	$\partial^2 u/\partial \theta^2$
u_{r_1}, u_{r_4}	gradient of u taken at midpoint of faces 1 and 4 of curvilinear cube
$u_{\theta_2}, u_{\theta_5}$	gradient of u taken at midpoint of faces 2 and 5 of curvilinear cube
V	volume
V_i, W_i, Z_i	matrices
ϵ	dielectric constant, F/M
ϕ	azimuthal angle in polar coordinates
$\vec{\phi}_1$	unit ϕ vector in spherical coordinates
ρ	volume charge density, C/m^3
θ	polar angle in spherical coordinates
$\bar{\theta}$	average value of θ
$\vec{\theta}_1$	unit θ vector in spherical coordinates
θ_2, θ_5	average values of θ at midpoint curvilinear cube faces 2 and 5
∇	vector differential operator
∇^2	Laplacian operator

Subscripts:

i	i^{th} ring
j	j^{th} (θ = constant) ray
l	upper limit of i
m	upper limit of j
n	order of block tridiagonal matrix

STATEMENT OF PROBLEM

The electric potential u satisfies Poisson's equation

$$\nabla^2 u = - \frac{\rho}{\epsilon} \tag{1}$$

where ρ is the space charge density and ϵ is the dielectric constant or permitivity. In spherical coordinates equation (1) is written

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \phi^2} = - \frac{\rho}{\epsilon} \tag{2}$$

or

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \phi^2} = - \frac{\rho}{\epsilon} \tag{2a}$$

Azimuthal symmetry is assumed in this problem ($\partial u/\partial \phi = 0$); therefore, the fifth term vanishes. The particular regions of interest are bounded by the coordinates $r \leq 1$

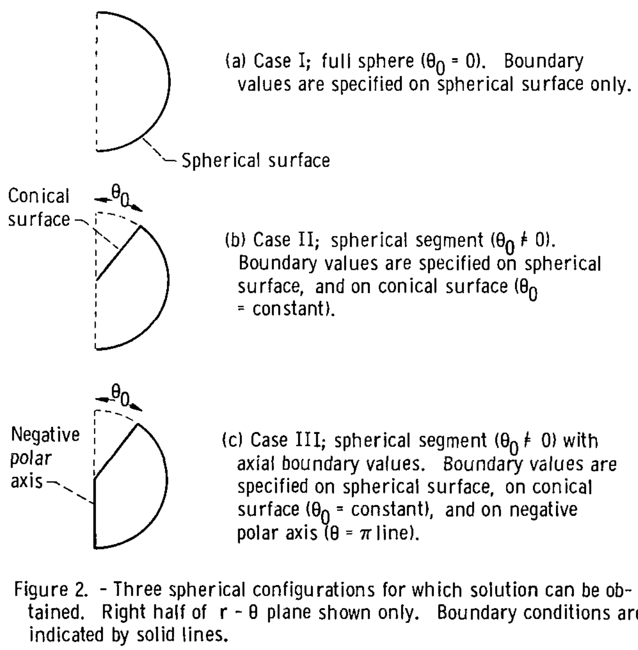


Figure 2. - Three spherical configurations for which solution can be obtained. Right half of r - θ plane shown only. Boundary conditions are indicated by solid lines.

and $0 \leq \theta_0 < \theta \leq \pi$. These regions are shown in figure 2 and are described as follows:

Case I ($\theta_0 = 0$): The volume is a full sphere and boundary conditions are prescribed on the surface only.

Case II ($\theta_0 \neq 0$): The volume is a spherical segment, and additional boundary values are specified on the conical surface (θ_0 = constant).

Case III: This case is identical to case II, with the additional option of specifying boundary values along the negative polar axis ($\theta = \pi$). Maximum allowable $r - \theta$ mesh size is 15 by 25. Mesh size and spacing is chosen by the user.

The mesh for the full sphere configuration was composed of (1) one center point, 15 interior r = constant rings, and the outer or surface boundary ring; (2) twenty-five θ = constant lines.

METHOD OF SOLUTION

Of the various methods known for solving this type problem, the one presented here requires the least computing time and storage.

These advantages are a consequence of ordering the mesh points in such a way as to produce a block tridiagonal matrix. The particular properties of this matrix are then used to develop a direct (noniterative) solution.

Derivation of Finite Difference Equation

We will now derive the finite difference equation for a general interior point by applying Gauss' divergence theorem to Poisson's equation in the cell. The resulting derivatives are then approximated by using a linear combination of each mesh point and its four neighbors.

Poisson's equation

$$\nabla^2 u = f \tag{3}$$

may be written for any volume V such that

$$\int_V \nabla^2 u \, dV = \int_V f \, dV \tag{4}$$

By the divergence theorem

$$\int_V \nabla \cdot \nabla u \, dV = \int_V \vec{n} \cdot \nabla u \, dS \tag{5}$$

where $\vec{n}$ is the outward unit normal and S is the surface of V. Equation (4) may now be rewritten as

$$\int_S \vec{n} \cdot \nabla u \, dS = \int_V f \, dV \tag{6}$$

Now consider a typical mesh point in spherical coordinates (r, θ, ϕ) enclosed by an

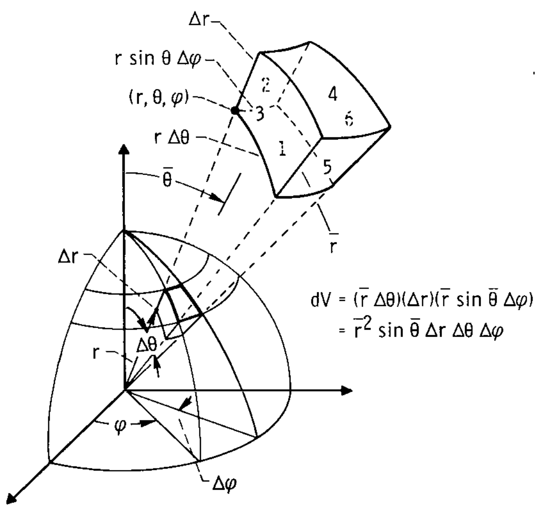


Figure 3. - Curvilinear element of volume in spherical coordinates.

element of volume as shown in figure 3. The surface of this cell is made up of three parts $S_r + S_\theta + S_\phi$, where S_r denotes left and right faces 1 and 4, S_θ denotes top and bottom faces 2 and 5, S_ϕ denotes front and rear faces 3 and 6. The area formulas of these faces and the volume of the element are then

$$
\left.\begin{aligned}
\Delta S_r &= r^2 \sin\theta\, \Delta\theta\, \Delta\phi \\
\Delta S_\theta &= r \sin\theta\, \Delta r\, \Delta\phi \\
\Delta S_\phi &= r\, \Delta r\, \Delta\theta \\
dV &= (r \sin\theta\, \Delta\phi)(r\, \Delta\theta)(\Delta r) \\
&= r^2 \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi
\end{aligned}\right\} \tag{7}
$$

Equation (6) may now be approximated by the volume bounded by this cell in the following manner:

$$
\nabla u = \vec{r}_1 u_r + \frac{1}{r}\vec{\theta}_1 u_\theta + \frac{1}{r \sin\theta}\vec{\phi}_1 u_\phi \tag{8}
$$

where $\vec{r}_1$, $\vec{\theta}_1$, and $\vec{\phi}_1$ are unit vectors in spherical coordinates. Therefore,

$$
\vec{n} \cdot \nabla u = \left\{\begin{array}{l}
- u_r \text{ on face } 1 \\
+ u_r \text{ on face } 4 \\
- \dfrac{u_\theta}{r} \text{ on face } 2 \\
+ \dfrac{u_\theta}{r} \text{ on face } 5 \\
- \dfrac{u_\phi}{r \sin\theta} \text{ on face } 3 \\
+ \dfrac{u_\phi}{r \sin\theta} \text{ on face } 6
\end{array}\right\} \tag{9}
$$

The ϕ terms in expressions (8) and (9) will be neglected since $u_\phi = 0$ in this problem.

The following terms are now defined:

$$\left.\begin{aligned}
\Delta S_{r_1} &= r_1^2 \sin\theta\ \Delta\theta\ \Delta\phi = \text{area of face 1} \\
\Delta S_{r_4} &= r_4^2 \sin\theta\ \Delta\theta\ \Delta\phi = \text{area of face 4} \\
\Delta S_{\theta_2} &= r \sin\theta_2\ \Delta r\ \Delta\phi = \text{area of face 2} \\
\Delta S_{\theta_5} &= r \sin\theta_5\ \Delta r\ \Delta\phi = \text{area of face 5}
\end{aligned}\right\} \tag{10}$$

Substituting equations (8) and (9) into equation (6), we obtain

$$u_{r_4} \Delta S_{r_4} - u_{r_1} \Delta S_{r_1} + \frac{1}{r} u_{\theta_5} \Delta S_{\theta_5} - \frac{1}{r} u_{\theta_2} \Delta S_{\theta_2} = f \cdot dV \tag{11}$$

where

$$\left.\begin{aligned}
u_{r_1} &= \left.\frac{\partial u}{\partial r}\right|_{r=r_1} \\
u_{r_4} &= \left.\frac{\partial u}{\partial r}\right|_{r=r_4} \\
u_{\theta_2} &= \left.\frac{\partial u}{\partial \theta}\right|_{\theta=\theta_2} \\
u_{\theta_5} &= \left.\frac{\partial u}{\partial \theta}\right|_{\theta=\theta_5}
\end{aligned}\right\} \tag{12}$$

Figure 4 shows that the area approximations in equation (11) may be improved by using the actual dimensions of the cell.

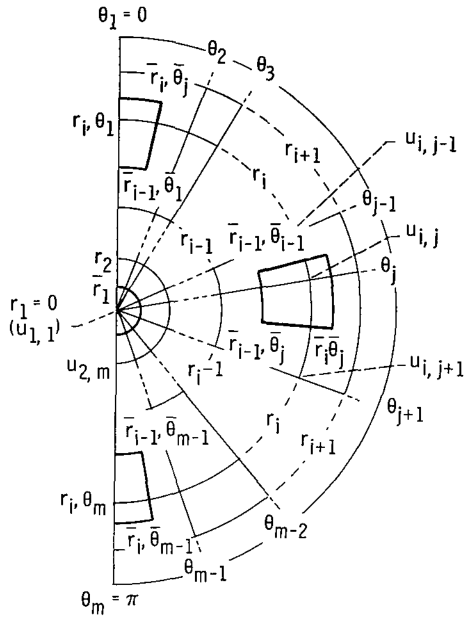


Figure 4. - Typical portions of mesh arrangement in r - θ plane. Heavy lines represent front face of curvilinear element of volume enclosing mesh points. $(1 \leq i \leq l, 1 \leq j \leq m)$.

$$
\left.
\begin{aligned}
\Delta S_{r_1} &= \frac{\bar{r}_{i-1}^2(\sin \bar{\theta}_j + \sin \bar{\theta}_{j-1})(\bar{\theta}_j - \bar{\theta}_{j-1})\Delta\phi}{2} \\
\Delta S_{r_4} &= \frac{\bar{r}_i^2(\sin \bar{\theta}_j + \sin \bar{\theta}_{j-1})(\bar{\theta}_j - \bar{\theta}_{j-1})\Delta\phi}{2} \\
\Delta S_{\theta_2} &= \frac{(\bar{r}_{i-1} \sin \bar{\theta}_{j-1} + \bar{r}_i \sin \bar{\theta}_{j-1})}{2} (\bar{r}_i - \bar{r}_{i-1})\Delta\phi \\
\Delta S_{\theta_5} &= \frac{(\bar{r}_{i-1} \sin \bar{\theta}_j + \bar{r}_i \sin \bar{\theta}_j)}{2} (\bar{r}_i - \bar{r}_{i-1})\Delta\phi
\end{aligned}
\right\} \qquad (13a)
$$

where

$$1 \leq i \leq l$$

$$1 \leq j \leq m$$

Similarly, the element of volume as originally expressed in equation (7) may now be refined as

$$dV = (\bar{r}_i - \bar{r}_{i-1})\left[\frac{\bar{r}_{i-1}(\bar{\theta}_j - \bar{\theta}_{j-1}) + \bar{r}_i(\bar{\theta}_j - \bar{\theta}_{j-1})}{2}\right]$$

$$\times\left[\frac{\bar{r}_i \sin \bar{\theta}_j + \bar{r}_i \sin \bar{\theta}_{j-1} + \bar{r}_{i-1} \sin \bar{\theta}_j + \bar{r}_{i-1} \sin \bar{\theta}_{j-1}}{4}\right]\Delta\phi \tag{13b}$$

Also, the four derivatives in equation (11) may be expressed as

$$\left.\begin{aligned}
u_{r_1} &= -\frac{(u_{i-1,j} - u_{i,j})}{\Delta r_{i-1}} \\
u_{r_4} &= \frac{(u_{i+1,j} - u_{i,j})}{\Delta r_i} \\
\frac{u_{\theta_2}}{\bar{r}} &= -\frac{(u_{i,j-1} - u_{i,j})}{(r_i \Delta\theta_{j-1})} \\
\frac{u_{\theta_5}}{\bar{r}} &= \frac{(u_{i,j+1} - u_{i,j})}{(r_i \Delta\theta_j)}
\end{aligned}\right\} \tag{13c}$$

Finally, by substituting equations (13a), (13b), and (13c) in equation (11) and dividing through by $\Delta\phi$, we obtain the finite difference equation for the typical interior mesh point.

$$
\left(\frac{u_{i+1,j} - u_{i,j}}{\Delta r_i}\right) \bar{r}_i(\bar{\theta}_j - \bar{\theta}_{j-1})\left(\frac{\bar{r}_i \sin \bar{\theta}_j + \bar{r}_i \sin \bar{\theta}_{j-1}}{2}\right)
$$

$$
+ \left(\frac{u_{i-1,j} - u_{i,j}}{\Delta r_{i-1}}\right) \bar{r}_{i-1}(\bar{\theta}_j - \bar{\theta}_{j-1})\left(\frac{\bar{r}_{i-1} \sin \bar{\theta}_j + \bar{r}_{i-1} \sin \bar{\theta}_{j-1}}{2}\right)
$$

$$
+ \left(\frac{u_{i,j+1} - u_{i,j}}{r_1 \Delta \theta_j}\right)(\bar{r}_i - \bar{r}_{i-1})\left(\frac{\bar{r}_i \sin \bar{\theta}_j + \bar{r}_{i-1} \sin \bar{\theta}_j}{2}\right)
$$

$$
+ \left(\frac{u_{i,j-1} - u_{i,j}}{r_i \Delta \theta_{j-1}}\right)(\bar{r}_i - \bar{r}_{i-1})\left(\frac{\bar{r}_i \sin \bar{\theta}_{j-1} + \bar{r}_{i-1} \sin \bar{\theta}_{j-1}}{2}\right)
$$

$$
= (\bar{r}_i - \bar{r}_{i-1})\left[\frac{\bar{r}_i(\bar{\theta}_j - \bar{\theta}_{j-1}) + \bar{r}_{i-1}(\bar{\theta}_j - \bar{\theta}_{j-1})}{2}\right]
$$

$$
\times \left(\frac{\bar{r}_i \sin \bar{\theta}_j + \bar{r}_i \sin \bar{\theta}_{j-1} + \bar{r}_{i-1} \sin \bar{\theta}_j + \bar{r}_{i-1} \sin \bar{\theta}_{j-1}}{4}\right) \cdot f
\tag{14a}
$$

where

$$2 \leq i \leq l - 1$$

$$2 \leq j \leq m$$

and where

$$
f = -\frac{\rho}{\epsilon}
\tag{14b}
$$

Reformulation of Finite Difference Equation

To simplify these expressions, we now use the following notation. Let

$$
\left.
\begin{aligned}
L_{i,j} &= \bar{r}_{i-1}(\bar{\theta}_j - \bar{\theta}_{j-1}) \frac{(\bar{r}_{i-1} \sin \bar{\theta}_j + \bar{r}_{i-1} \sin \bar{\theta}_{j-1})}{(2\ \Delta r_{i-1})} \\
R_{i,j} &= \bar{r}_i(\bar{\theta}_j - \bar{\theta}_{j-1}) \frac{(\bar{r}_i \sin \bar{\theta}_j + \bar{r}_i \sin \bar{\theta}_{j-1})}{(2\ \Delta r_i)} \\
T_{i,j} &= (\bar{r}_i - \bar{r}_{i-1}) \frac{(\bar{r}_i \sin \bar{\theta}_{j-1} + \bar{r}_{i-1} \sin \bar{\theta}_{j-1})}{(2 r_i\ \Delta\theta_{j-1})} \\
B_{i,j} &= (\bar{r}_i - \bar{r}_{i-1}) \frac{(\bar{r}_i \sin \bar{\theta}_j + \bar{r}_{i-1} \sin \bar{\theta}_j)}{(2 r_i\ \Delta\theta_j)} \\
F_{i,j} &= (\bar{r}_i - \bar{r}_{i-1}) \left[\frac{\bar{r}_i(\bar{\theta}_j - \bar{\theta}_{j-1}) + \bar{r}_{i-1}(\bar{\theta}_j - \bar{\theta}_{j-1})}{2}\right] \\
&\qquad \times \left[\frac{\bar{r}_i \sin \bar{\theta}_j + \bar{r}_i \sin \bar{\theta}_{j-1} + \bar{r}_{i-1} \sin \bar{\theta}_j + \bar{r}_{i-1} \sin \bar{\theta}_{j-1}}{4}\right]
\end{aligned}
\right\} \quad (15)
$$

where

$$2 \leq i \leq l-1$$

$$2 \leq j \leq m$$

Substituting these expressions in equation (14a) then yields

$$L_{i,j}(u_{i-1,j} - u_{i,j}) + R_{i,j}(u_{i+1,j} - u_{i,j}) + T_{i,j}(u_{i,j-1} - u_{i,j}) + B_{i,j}(u_{i,j+1} - u_{i,j}) = F_{i,j}(f) \quad (16)$$

or

$$L_{i,j}u_{i-1,j} + R_{i,j}u_{i+1,j} + T_{i,j}u_{i,j-1} + B_{i,j}u_{i,j+1} - u_{i,j}(L_{i,j} + R_{i,j} + T_{i,j} + B_{i,j}) = f \cdot F_{i,j} \quad (17)$$

Finally,

$$D_{i,j}u_{i,j} + L_{i,j}u_{i-1,j} + R_{i,j}u_{i+1,j} + T_{i,j}u_{i,j-1} + B_{i,j}u_{i,j+1} = F_{i,j}f \tag{18a}$$

where

$$D_{i,j} = -\left(L_{i,j} + R_{i,j} + T_{i,j} + B_{i,j}\right) \tag{18b}$$

These L, R, T, B, and D expressions refer to the coefficients associated with any set of left, right, top, bottom and central mesh points as shown in figure 5. Because of this point configuration, equation (18a) is called a five-point formula. This equation is valid for all points except those on the polar axis. We will now discuss these exceptions by considering three special conditions; (1) the center point, (2) points on the $\theta = 0$ positive polar axis, and (3) points on the $\theta = \pi$ negative polar axis.

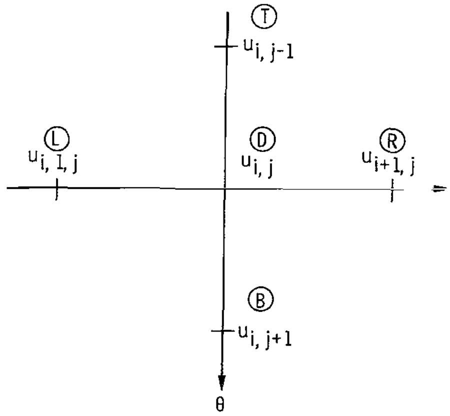


Figure 5. - Five-point mesh configuration.

(1) Center point ($i = 1$, $1 \leq j \leq m$): In this case the cell simply becomes a sphere; and equation (18a) uses D- and R-coefficients only and becomes

$$D_{1,1}u_{1,1} + \sum_{j=1}^{m} R_{1,j}u_{2,j} = \sum_{j=1}^{m} F_{1,j} \cdot f \tag{19a}$$

(2) Points on the $\theta = 0$ axis ($2 \leq i \leq l-1, j = 1$): Setting $\overline{\theta}_0 = \theta_1$, equation (18a) reduces to

$$D_{i,1}u_{i,1} + L_{i,1}u_{i-1,1} + R_{i,1}u_{i+1,1} + 2B_{i,1}u_{i,2} = F_{i,1} \quad f \tag{19b}$$

(3) Points on the $\theta = \pi$ axis $(2 \leq i \leq l-1, j = m)$: With $\theta_{m+1} = \theta_m$, equation (18a) reduces to

$$D_{i,m}u_{i,m} + L_{i,m}u_{i-1,m} + R_{i,m}u_{i+1,m} + 2T_{i,m}u_{i,m-1} = F_{i,m} \cdot f \tag{19c}$$

Notice that in the last two cases, the cell becomes the frustum of a spherical cone.

Summarizing then, for each interior mesh point (r_i, θ_j) where $u_{i,j}$ is unknown, the finite difference expression is derived in terms of its four adjacent mesh points. Notice that three kinds of cells were used:

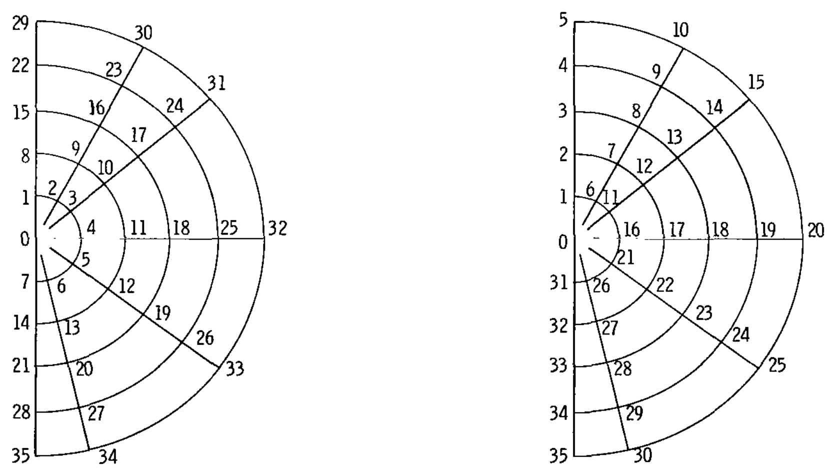
(1) The curvilinear cube enclosing all mesh points interior to the sphere but not on the axis,

(2) The sphere enclosing the center point,

(3) The frustum of a spherical cone enclosing all axial points except the central point.

Ordering of Mesh Points

In this section we will explore how the ordering of mesh points affects the form of the coefficient matrix. Let us consider first case I, the full sphere configuration. There are two natural ways of point ordering, ring by ring or ray by ray. Both methods are shown below in figure 6 for a 35-point sample (5 r-rings and 7 θ-rays).



(a) Ring-by-ring ordering.

(b) Ray-by-ray ordering.

Figure 6. - Example of mesh point ordering.

The ring-by-ring ordering shown in figure 4 has considerable advantages. At mesh point 0, equation (19a) would use points 0, 1, 2, ..., 7. At mesh point 1, equation (19b) would use points 0, 8, 2, and 1 for the L, R, B, and D coefficients, respectively. Similarly, for mesh point 2, equation (18a) would use points 0, 9, 1, 3, and 2 for the L, R, T, B, and D coefficients, respectively. This method is then continued for points 3 to 6. For mesh point 7, equation (19c) would use points 0, 6, 7, and 14. This process is then continued clockwise and outward for each ring until the finite difference equation is written at all mesh points. The resultant coefficient matrix is shown in figure 7. Note

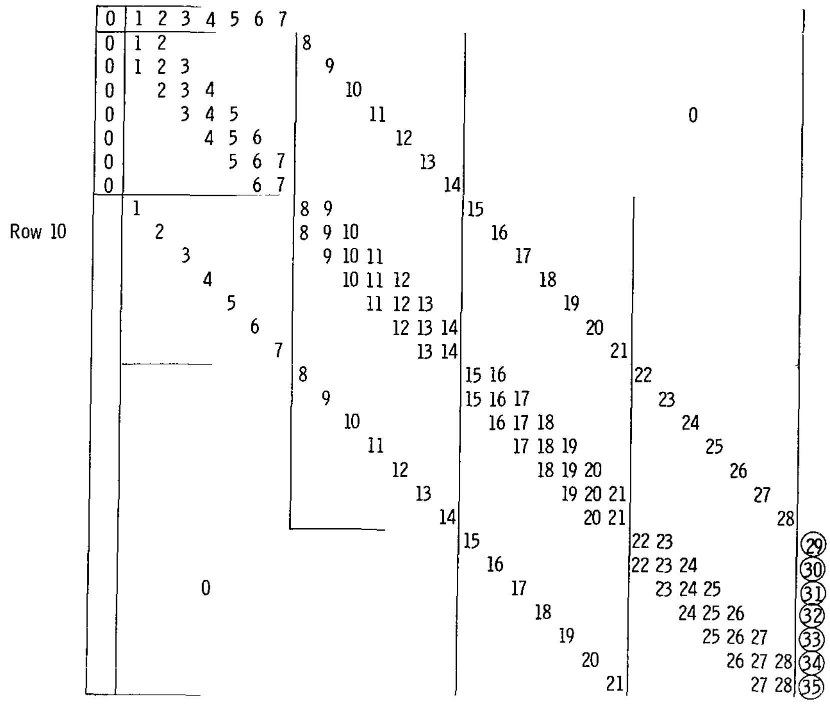


Figure 7. - Example of block tridiagonal coefficient matrix resulting from applying the five-point finite difference equation at each interior mesh point of the full sphere configuration. Note that the diagonal blocks are tridiagonal matrices, and the off-diagonal blocks are simply diagonal matrices. Circled numbers represent coefficients of the surface boundary values. (The elements of any particular row are formed by L . . . TDB . . . R coefficients. For example, the elements of row 10 (2 . . . 8, 9, 10 . . . 16) are formed by the left, top, central, bottom and right coefficients associated with mesh point 9 in fig. 4).

that the partitioning of the rows and columns of the matrix in a 1, 7, 7, 7, 7 fashion results in nonzero elements lying in blocks that are zero except for the diagonal blocks, the superdiagonal blocks, and the subdiagonal blocks. We define this matrix as "block tridiagonal."

The other method of point ordering (ray-by-ray) shown in figure 6 would not yield a matrix with this block tridiagonal property. This is illustrated in figure 8. In a sim-

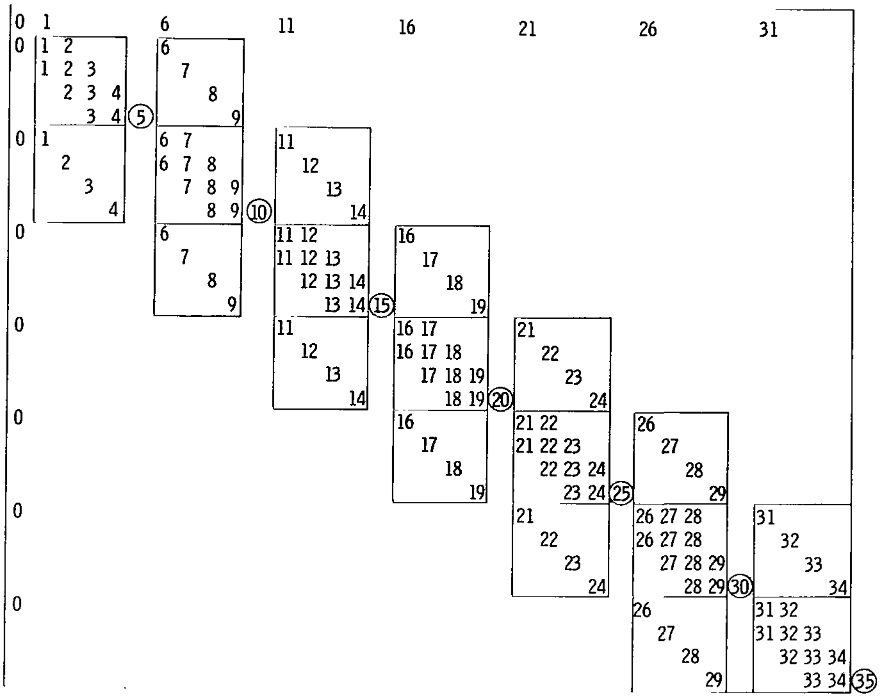


Figure 8. - Coefficient matrix resulting from improper choice of mesh point ordering. Notice the outriding coefficients in row 1 and column 1. Circled numbers represent coefficients of surface boundary values.

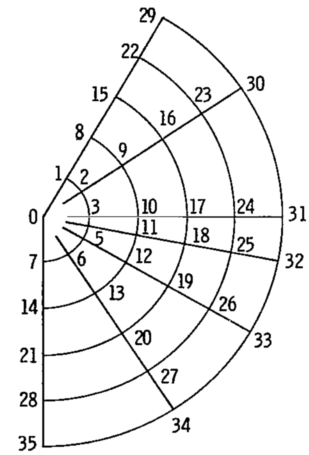


Figure 9. - Example of ordering of mesh points for spherical segment (cases II and III) Boundary values are specified on the spherical surface by points 29 to 35 and on the conical surface (θ_0 = constant surface) by points 0, 1, 8, 15, 22, and 29. Optional boundary values on the negative polar axis ($\theta = \pi$ line) are specified by points 0, 7, 14, 21, 28, and 35.

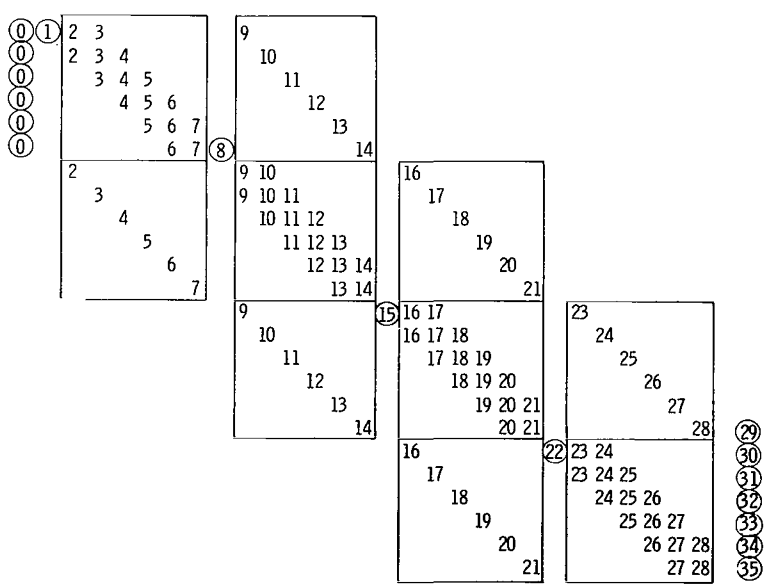


Figure 10. - Example of block tridiagonal coefficient matrix generated from the mesh points of the unknowns in the spherical segment configuration. Circled numbers represent coefficients for the boundary values.

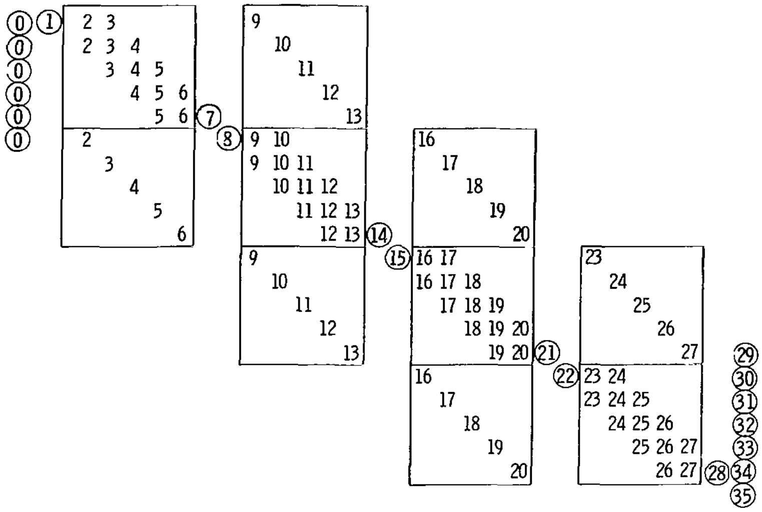


Figure 11. - Example of block tridiagonal coefficient matrix resulting from applying finite difference equation to mesh points of each unknown in example of spherical segment with axial values. Circled points represent coefficients for boundary values.

ilar fashion, the point ordering and matrix generation for cases II and III are shown in figures 9 to 11. Note that the block tridiagonal property persists throughout, the only change being that of a decrease in the order of the matrices as one moves from case to case.

Generation of Coefficient Matrix

We have just shown with a small sample mesh how proper ordering of points is necessary for the formation of the block tridiagonal matrix.

The corresponding coefficient matrix associated with case I, the full sphere configuration, will now be examined. Here, ring-by-ring ordering of the 15 by 25 r-θ mesh points yields a block tridiagonal matrix whose nonzero elements all lie in a band of width 51 around the principal diagonal. This was shown previously in figure 7.

It is possible at this time to write the matrix equation and, using an algorithm of the type given in reference 1, to obtain a solution. But this would necessitate working with a matrix requiring 19 125 storages (51 by 375). But, when a coefficient matrix exhibits the tridiagonal block property and, in addition, has off-diagonal blocks that are in themselves diagonal, then a modification of the algorithm enables one to solve the matrix equation at a considerable savings in storage. One needs to store only the inverse of the block matrices associated with the principal diagonal blocks. For the present mesh size, this would require only 9375 storages (15 by 625).

Note also that the matrix is symmetrical. This is clearly shown by figure 7 and equation (15) where

$$L_{i+1,j} = R_{i,j}$$

and

$$T_{i,j+1} = B_{i,j}$$

From this property we may now write

$$A_{i+1} = C_i$$

where the A's and C's are block matrices as defined in equation (20).

In this problem the five-point stencil previously shown in figure 5 produces a set of matrix equations $MZ = K$, where matrix M is block tridiagonal of the form

$$\left.\begin{aligned} M &= \begin{bmatrix} B_1 & C_1 & 0 & 0 \\ A_2 & B_2 & C_2 & \\ 0 & \ddots & \ddots & C_{n-1} \\ 0 & & A_n & B_n \end{bmatrix} \\ &\text{and} \\ K &= \begin{bmatrix} K_1 \\ K_2 \\ \cdot \\ \cdot \\ \cdot \\ K_n \end{bmatrix} \end{aligned}\right\} \tag{20}$$

whose elements are matrices. Note that the A's and C's are diagonal, the B's are tridiagonal symmetric, and K is simply a column vector.

Development of Algorithm

The matrix equation can be solved directly by using the algorithm of reference 1.

Matrices $W_1, W_2, \ldots, W_{n-1}$ are calculated from

$$\left.\begin{aligned} W_1 &= B_1^{-1}C_1 \\ W_i &= (B_i - A_iW_{i-1})^{-1}C_i \qquad 2 \le i \le (n-1) \end{aligned}\right\} \tag{21a}$$

Vectors $G_1, G_2, \ldots., G_n$ are calculated from

$$\left.\begin{aligned} G_1 &= B_1^{-1}K_1 \\ G_i &= (B_i - A_iW_{i-1})^{-1}(K_i - A_iG_{i-1}) \qquad 2 \le i \le n \end{aligned}\right\} \tag{21b}$$

Then the vector components Z_i of the solution of $MZ = K$ are given recursively by

$$\left.\begin{aligned} Z_n &= G_n \\ Z_i &= G_i - W_iZ_{i+1} \qquad (n-1) \ge i \ge 1 \end{aligned}\right\} \tag{21c}$$

The algorithm may be modified as follows. Rewrite equation (21a) as

$$(B_i - A_iW_{i-1})^{-1} = W_iC_i^{-1} \tag{22a}$$

Now define

$$V_i = W_iC_i^{-1} \tag{22b}$$

then

$$W_{i-1} = V_{i-1}C_{i-1} \tag{22c}$$

Since by symmetry $A_i = C_{i-1}$, this may be rewritten as

$$W_{i-1} = V_{i-1}A_i \tag{22d}$$

By definition $V_i = (B_i - A_iW_{i-1})^{-1}$, then

$$V_i = (B_i - A_iV_{i-1}A_i)^{-1} \tag{22e}$$

Using equation (22e), the algorithm may now be rewritten as

$$\left.\begin{aligned} V_1 &= B_1^{-1} \\ V_i &= (B_i - A_i V_{i-1} A_i)^{-1} \qquad 2 \le i \le n \end{aligned}\right\} \tag{23a}$$

$$\left.\begin{aligned} G_1 &= B_1^{-1} K_1 \qquad \text{(Unchanged)} \\ G_i &= V_i (K_i - A_i G_{i-1}) \qquad 2 \le i \le n \end{aligned}\right\} \tag{23b}$$

$$\left.\begin{aligned} Z_n &= G_n \qquad \text{(Unchanged)} \\ Z_i &= G_i - V_i A_{i+1} Z_{i+1} \qquad (n-1) \ge i \ge 1 \end{aligned}\right\} \tag{23c}$$

Notice that two multiplication operations have been eliminated, one in the calculation of V_i, and one in the calculation of G_i. For the full sphere, B_1 is a scalar, and A_2 is an NT by 1 column matrix (L-coefficients of innermost ring). The V_i are full and symmetric, and these are all that must be stored (15 by 625 storages).

The following section describes the FORTRAN program written from the algorithm just developed.

THE FORTRAN PROGRAM

General Description

The program consists of nine subroutines:

(1) POISS1 - generates $\bar{r}$ and $\bar{\theta}$ mesh arrays and sets all switches required by the spherical geometry specified.

(2) SPHER1 - generates all coefficient matrices required for case I, the full sphere configuration.

(3) SPHER2 - uses the algorithm to calculate the solution for case I.

(4) SEG1 - generates all coefficient matrices required for either the case II or case III configuration

(5) SEG2 - uses the algorithm to calculate the solution for either case II or case III.

(6) FACTOR - performs matrix factorization before calculating inverses.

(7) INVERT - performs matrix inversions.

(8) AE - calculates coefficients associated with each interior mesh point.

(9) FE - calculates coefficients associated with each value of the source term array.

The FORTRAN program is listed in appendix B and flow charts in appendix D. All FORTRAN symbols used are defined in appendix C. The program will return a solution in about 6 or 7 seconds and requires approximately 10 000 storages.

Using the Program

Using the subroutine is easy. First, the appropriate quantities are stored in

AVS (logical constant), NR, NT, X1, RA(NR), TA(NT), and X(NT,NR-1)

Subroutine POISS1 is called. Then subroutine POISS2 is called, and the solution is returned in the X-array. This procedure will now be described in detail.

Regardless of which case is being run, the main calling program must contain the following COMMON card

```
COMMON/BLOCK1/NR,NT,RA(17),TA(25),X(25,16),X1,AVS
```

and the following definitions:

NR	number of r mesh lines, including the center point (maximum value, 17)
NT	number of θ mesh lines (maximum value, 25)
RA(IR)	r mesh values, $1 \leq IR \leq NR$ (note that RA(1) = 0)
TA(IT)	θ mesh values, $1 \leq IT \leq NT$ (note that TA(NT) = π)
X(IT,NR-1)	surface boundary values, $1 \leq IT \leq NT$

In addition, the main calling program must provide the following data:

Case I, $\theta = 0$ (full sphere configuration): Load space charge terms into X(IT,IR), $1 \leq IT \leq NT$, $1 \leq IR \leq NR-2$

Call POISS1 then POISS2. The solution is then returned in the X-array.

Case II, $\theta \neq 0$ (spherical cone; no axial values):

(a) Load center point value into X1.

(b) Load additional boundary values from the conical surface (θ = constant) into X(1,IR) ($1 \leq IR \leq NR-2$).

(c) Load space charge terms into X(IT,IR) ($2 \leq IT \leq NT$; $1 \leq IR \leq NR-2$).

(d) Set AVS to false.

Call POISS1, then POISS2. The solution is then returned in the X-array.

Case III, $\theta \neq 0$ (spherical cone with axial values):

(a) Load center point value into X1

(b) Load additional boundary values from the conical surface (θ = constant) into X(1,IR) ($1 \leq IR \leq NR-2$).

(c) Load axial boundary values into X(NT,IR) ($1 \leq IR \leq NR-2$).

(d) Load space charge terms into X(IT,IR) ($2 \leq IT \leq NT-1$, $1 \leq IR \leq NR-2$).

(e) Set AVS to true.

Call POISS1, then POISS2. The solution is then returned in the X-array. Note that in all cases the solution is returned in the X-array, but the boundary values remain undisturbed. When repeated solutions are sought for the same mesh but for different space charge terms, POISS1 need be called only once.

A sample test run is given in appendix A which illustrates the running of each of the three cases described.

CONCLUDING REMARKS

We have described a numerical method and computer program for solving Poisson's equation in axisymmetric regions of a sphere of two geometries:

(1) A full sphere with boundary values on the surface only

(2) A spherical cone with additional boundary values on the conical surface, and, optionally, along the negative polar axis.

In developing the numerical method, a curvilinear cell was used at each mesh point to derive the finite difference analog to Poisson's equation. The mesh points were then ordered in such a manner so as to produce a block tridiagonal symmetric coefficient matrix. The particular properties of this type of matrix were then used to write an algorithm from which a computer program was developed. The program uses a direct, noniterative solution and requires minimal storage of the requisite matrices.

Two types of flexibility were built into the program: variable spherical geometry and irregular mesh spacing in two directions.

It should be noted that the cell approach presents a simple way to avoid numerical difficulties in regions where singularities might be present. For example, in this problem such a region would be along the polar axis. The numerical method is also general enough for easy extension to other equations and other geometries.

The program is easy to use and general enough to be used as a black box in any situation where a solution to Poisson's equation is needed in spherical regions.

For a mesh with a maximum allowable number of 376 points, the program returns a solution in 6 to 7 seconds (IBM DCS 7094/7044 computer) and requires an approximately 10 000-word storage.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, April 14, 1971,
129-04.

APPENDIX A

SAMPLE PROBLEM

A potential distribution was assumed to be of the form

$$u = r^3 \cos \theta \tag{A1}$$

Therefore,

$$\left.\begin{aligned} u_r &= 3r^2 \cos \theta \\ u_{rr} &= 6r \cos \theta \\ u_\theta &= -r^3 \sin \theta \\ u_{\theta\theta} &= -r^3 \cos \theta \end{aligned}\right\} \tag{A2}$$

Substituting these quantities into equation (2a), the expression for the right-hand side is obtained:

$$6r \cos \theta + \frac{2}{r} 3r^2 \cos \theta + \frac{1}{r^2} (-r^3 \cos \theta) + \frac{\cot \theta}{r^2} (-r^3 \sin \theta) = 10r \cos \theta \tag{A3}$$

For all three cases, NR = 17, NT = 25, and the RA array was 0, 0.2, 0.3, 0.39, 0.48, 0.56, 0.63, 0.70, 0.76, 0.81, 0.86, 0.90, 0.93, 0.96, 0.98, 0.99, and 1.00.

The TA array varied for all three cases in the following manner:

Case I ($\theta_0 = 0^\circ$) - TA = 0, 14, 28, 41, 54, 66, 77, 88, 98, 108, 117, 125, 133, 140, 147, 153, 158, 163, 167, 171, 174, 176, 178, 179, 180.

Case II ($\theta_0 = 45^\circ$) - TA = 45, 56, 66, 76, 86, 95, 103, 111, 119, 126, 133, 139, 145, 150, 155, 160, 164, 168, 171, 174, 176, 177, 178, 179, 180.

Case III ($\theta_0 = 60^\circ$) - TA = 60, 70, 79, 88, 96, 104, 112, 119, 126, 132, 138, 144, 149, 154, 158, 162, 166, 169, 172, 174, 176, 177, 178, 179, 180.

Once these values are stored, the values on the right-hand side are calculated from equation (A3) and stored in X(IT, NR-1) (where IT = 1, 2, ... , 25 and NR = 2, 3, ... , 16). The following sets of boundary values are now calculated from equation (A1). For cases I, II, and III, the surface boundary values (r = 1) are calculated and stored in X(IT, 16) (where IT = 1, 2, ... , 25). For cases II and III, the conical surface

(θ = constant) boundary values are calculated and stored in X(1, IR) (where IR = 1, 2, . . , 16). For case III only, the negative axial ($\theta = \pi$) boundary values are calculated and stored in X(25, IR) (where IR = 1, 2, ... , 16).

POISS1 is then called, followed by the final call, POISS2. The exact answers are calculated using equation (A1) and compared with those calculated by the subroutine. These residuals are also printed out for each case. Note that the largest residuals occur in those areas where the mesh spacing is coarsest.

The FORTRAN program and output follow:

```
$IBFTC TEST
      COMMON/BLOCK1/NR,NT,RA(17),TA(25),X(25,16),X1,AVS
      LOGICAL AVS
      NAME LIST/DATA/ TA,RA,NR,NT
C
   14 FORMAT(1X,F6.2,16F7.2)
   15 FORMAT(6H THETA,3X,16(F5.3,2X))
   16 FORMAT(11H1RESIDUALS-,41X,3HRHO)
   17 FORMAT(14H0AXIAL VALUES-)
   18 FORMAT(4X,1HR,8X,1HV)
   19 FORMAT(2X,F5.3,3X,F6.3)
   20 FORMAT(8H1OUTPUT-,52X,3HRHO)
   21 FORMAT(14H0EXACT ANSWERS,45X,3HRHO)
   22 FORMAT(1X,F6.2,16F7.4)
   23 FORMAT(7H0INPUT-,53X,3HRHO)
   24 FORMAT(1H1)
   25 FORMAT(55H)
   26 FORMAT(6H XCTR=,F5.2)
C
      CVF=57.29578
      DO 13 L=1,3
      WRITE(6,24)
      READ(5,25)
      WRITE(6,25)
      READ(5,25)
      WRITE(6,25)
      READ(5,DATA)
      AVS=.FALSE.
      NRM1=NR-1
C
C         CONVERT THETA ARRAY TO RADIANS.
      DO 1 J=1,NT
    1 TA(J)=TA(J)/CVF
C
C         GENERATE RIGHT-HAND SIDE.
      DO 2 IT=1,NT
       TH=TA(IT)
       CT=COS(TH)
      DO 2 IR=1,NRM1
    2 X(IT,IR)=10.*RA(IR+1)*CT
      X1=0.0
C
```

```
C         CALCULATE SURFACE BOUNDARY VALUES.
      DO 3 IT=1,NT
    3 X(IT,16)=COS(TA(IT))
      IF(L.EQ.1) GO TO 7
C
C         CALCULATE RADIAL BOUNDARY VALUES.
      CT=COS(TA(1))
      DO 4 IR=1,NRM1
    4 X(1,IR)=(RA(IR+1)**3)*CT
      IF(L.EQ.2) GO TO 7
C
C         CALCULATE/PRINT AXIAL BOUNDARY VALUES.
      AVS=.TRUE.
      CT=COS(TA(NT))
      DO 5 IR=1,NRM1
      R3=RA(IR+1)**3
    5 X(NT,IR)=R3*CT
      WRITE(6,17)
      WRITE(6,18)
      WRITE(6,19) RA(1),X1
      DO 6 I=1,NRM1
      IP1=I+1
    6 WRITE(6,19) RA(IP1),X(NT,I)
    7 CALL POISS1
C
C         PRINT INPUT BLOCK.
      WRITE(6,23)
      WRITE(6,15) (RA(I), I=2,NR)
      DO 8 J=1,NT
      THD=CVF*TA(J)
    8 WRITE(6,14) THD, (X(J,K-1), K=2,NR)
      CALL POISS2
C
C         PRINT OUTPUT BLOCK.
      WRITE(6,20)
      WRITE(6,15) (RA(I), I=2,NR)
      DO 9 J=1,NT
      THD=CVF*TA(J)
    9 WRITE(6,22) THD,(X(J,K-1),K=2,NR)
      WRITE(6,26) X1
C
C         CALCULATE AND PRINT EXACT ANSWERS.
      WRITE(6,21)
      WRITE(6,15) (RA(I), I=2,NR)
      DO 10 I=1,NRM1
      DO 10 J=1,NT
      R3=RA(I+1)**3
      CT=COS(TA(J))
C
C         RESIDUALS.
      DIMENSION RES(25,16)
      T1=R3*CT
      RES(J,I)=X(J,I)-T1
   10 X(J,I)=T1
      DO 11 J=1,NT
      THD=CVF*TA(J)
   11 WRITE(6,22) THD, (X(J,K-1), K=2,NR)
      WRITE(6,26) X1
C
```

```
C         PRINT RESIDUALS.
      WRITE(6,16)
      WRITE(6,15) (RA(I), I=2,NR)
      DO 12 J=1,NT
      THD=CVF*TA(J)
   12 WRITE(6,22) THD, (RES(J,K-1), K=2,NR)
   13 CONTINUE
C
      STOP
      END
```

TEST CASE 1, THETA0=0 DEGREES.
(FULL SPHERE).

INPUT-								RHO								
THETA	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
0.	2.00	3.00	3.90	4.80	5.60	6.30	7.00	7.60	8.10	8.60	9.00	9.30	9.60	9.80	9.90	1.00
14.00	1.94	2.91	3.78	4.66	5.43	6.11	6.79	7.37	7.86	8.34	8.73	9.02	9.31	9.51	9.61	0.97
28.00	1.77	2.65	3.44	4.24	4.94	5.56	6.18	6.71	7.15	7.59	7.95	8.21	8.48	8.65	8.74	0.88
41.00	1.51	2.26	2.94	3.62	4.23	4.75	5.28	5.74	6.11	6.49	6.79	7.02	7.25	7.40	7.47	0.75
54.00	1.18	1.76	2.29	2.82	3.29	3.70	4.11	4.47	4.76	5.05	5.29	5.47	5.64	5.76	5.82	0.59
66.00	0.81	1.22	1.59	1.95	2.28	2.56	2.85	3.09	3.29	3.50	3.66	3.78	3.90	3.99	4.03	0.41
77.00	0.45	0.67	0.88	1.08	1.26	1.42	1.57	1.71	1.82	1.93	2.02	2.09	2.16	2.20	2.23	0.22
88.00	0.07	0.10	0.14	0.17	0.20	0.22	0.24	0.27	0.28	0.30	0.31	0.32	0.34	0.34	0.35	0.03
98.00	-0.28	-0.42	-0.54	-0.67	-0.78	-0.88	-0.97	-1.06	-1.13	-1.20	-1.25	-1.29	-1.34	-1.36	-1.38	-0.14
108.00	-0.62	-0.93	-1.21	-1.48	-1.73	-1.95	-2.16	-2.35	-2.50	-2.66	-2.78	-2.87	-2.97	-3.03	-3.06	-0.31
117.00	-0.91	-1.36	-1.77	-2.18	-2.54	-2.86	-3.18	-3.45	-3.68	-3.90	-4.09	-4.22	-4.36	-4.45	-4.49	-0.45
125.00	-1.15	-1.72	-2.24	-2.75	-3.21	-3.61	-4.02	-4.36	-4.65	-4.93	-5.16	-5.33	-5.51	-5.62	-5.68	-0.57
133.00	-1.36	-2.05	-2.66	-3.27	-3.82	-4.30	-4.77	-5.18	-5.52	-5.87	-6.14	-6.34	-6.55	-6.68	-6.75	-0.68
140.00	-1.53	-2.30	-2.99	-3.68	-4.29	-4.83	-5.36	-5.82	-6.20	-6.59	-6.89	-7.12	-7.35	-7.51	-7.58	-0.77
147.00	-1.68	-2.52	-3.27	-4.03	-4.70	-5.28	-5.87	-6.37	-6.79	-7.21	-7.55	-7.80	-8.05	-8.22	-8.30	-0.84
153.00	-1.78	-2.67	-3.47	-4.28	-4.99	-5.61	-6.24	-6.77	-7.22	-7.66	-8.02	-8.29	-8.55	-8.73	-8.82	-0.89
158.00	-1.85	-2.78	-3.62	-4.45	-5.19	-5.84	-6.49	-7.05	-7.51	-7.97	-8.34	-8.62	-8.90	-9.09	-9.18	-0.93
163.00	-1.91	-2.87	-3.73	-4.59	-5.36	-6.02	-6.69	-7.27	-7.75	-8.22	-8.61	-8.89	-9.18	-9.37	-9.47	-0.96
167.00	-1.95	-2.92	-3.80	-4.68	-5.46	-6.14	-6.82	-7.41	-7.89	-8.38	-8.77	-9.06	-9.35	-9.55	-9.65	-0.97
171.00	-1.98	-2.96	-3.85	-4.74	-5.53	-6.22	-6.91	-7.51	-8.00	-8.49	-8.89	-9.19	-9.48	-9.68	-9.78	-0.99
174.00	-1.99	-2.98	-3.88	-4.77	-5.57	-6.27	-6.96	-7.56	-8.06	-8.55	-8.95	-9.25	-9.55	-9.75	-9.85	-0.99
176.00	-2.00	-2.99	-3.89	-4.79	-5.59	-6.28	-6.98	-7.58	-8.08	-8.58	-8.98	-9.28	-9.58	-9.78	-9.88	-1.00
178.00	-2.00	-3.00	-3.90	-4.80	-5.60	-6.30	-7.00	-7.60	-8.10	-8.59	-8.99	-9.29	-9.59	-9.79	-9.89	-1.00
179.00	-2.00	-3.00	-3.90	-4.80	-5.60	-6.30	-7.00	-7.60	-8.10	-8.60	-9.00	-9.30	-9.60	-9.80	-9.90	-1.00
180.00	-2.00	-3.00	-3.90	-4.80	-5.60	-6.30	-7.00	-7.60	-8.10	-8.60	-9.00	-9.30	-9.60	-9.80	-9.90	-1.00

OUTPUT-

THETA \ RHO	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
0.	0.0089	0.0282	0.0605	0.1114	0.1762	0.2504	0.3432	0.4390	0.5314	0.6360	0.7289	0.8043	0.8847	0.9412	0.9703	1.0000
14.00	0.0086	0.0274	0.0587	0.1081	0.1709	0.2430	0.3329	0.4260	0.5156	0.6171	0.7073	0.7804	0.8584	0.9132	0.9415	0.9703
28.00	0.0078	0.0249	0.0534	0.0983	0.1555	0.2210	0.3029	0.3875	0.4692	0.5615	0.6435	0.7101	0.7811	0.8310	0.8567	0.8829
41.00	0.0066	0.0212	0.0456	0.0840	0.1328	0.1889	0.2589	0.3312	0.4010	0.4799	0.5501	0.6070	0.6677	0.7103	0.7323	0.7547
54.00	0.0051	0.0164	0.0354	0.0653	0.1033	0.1470	0.2015	0.2578	0.3122	0.3737	0.4283	0.4727	0.5200	0.5532	0.5703	0.5878
66.00	0.0034	0.0113	0.0244	0.0451	0.0714	0.1016	0.1393	0.1783	0.2160	0.2585	0.2963	0.3270	0.3598	0.3828	0.3946	0.4067
77.00	0.0018	0.0061	0.0133	0.0248	0.0394	0.0561	0.0770	0.0986	0.1194	0.1429	0.1639	0.1809	0.1990	0.2117	0.2183	0.2250
88.00	0.0000	0.0007	0.0018	0.0036	0.0058	0.0085	0.0117	0.0151	0.0183	0.0220	0.0253	0.0280	0.0308	0.0328	0.0338	0.0349
98.00	-0.0016	-0.0043	-0.0088	-0.0158	-0.0248	-0.0351	-0.0480	-0.0613	-0.0741	-0.0886	-0.1015	-0.1120	-0.1232	-0.1310	-0.1350	-0.1392
108.00	-0.0031	-0.0091	-0.0191	-0.0348	-0.0548	-0.0777	-0.1063	-0.1359	-0.1644	-0.1967	-0.2254	-0.2486	-0.2734	-0.2909	-0.2998	-0.3090
117.00	-0.0045	-0.0132	-0.0278	-0.0509	-0.0803	-0.1140	-0.1560	-0.1995	-0.2414	-0.2889	-0.3310	-0.3652	-0.4017	-0.4273	-0.4405	-0.4540
125.00	-0.0056	-0.0166	-0.0351	-0.0642	-0.1013	-0.1439	-0.1970	-0.2520	-0.3049	-0.3649	-0.4181	-0.4614	-0.5075	-0.5398	-0.5565	-0.5736
133.00	-0.0065	-0.0197	-0.0416	-0.0763	-0.1204	-0.1710	-0.2342	-0.2995	-0.3625	-0.4338	-0.4972	-0.5486	-0.6034	-0.6419	-0.6617	-0.6820
140.00	-0.0073	-0.0221	-0.0467	-0.0856	-0.1352	-0.1920	-0.2630	-0.3364	-0.4072	-0.4872	-0.5584	-0.6161	-0.6777	-0.7210	-0.7433	-0.7660
147.00	-0.0080	-0.0241	-0.0511	-0.0937	-0.1480	-0.2102	-0.2879	-0.3683	-0.4458	-0.5334	-0.6113	-0.6746	-0.7420	-0.7893	-0.8138	-0.8387
153.00	-0.0084	-0.0256	-0.0543	-0.0995	-0.1572	-0.2233	-0.3058	-0.3912	-0.4736	-0.5667	-0.6495	-0.7166	-0.7883	-0.8386	-0.8645	-0.8910
158.00	-0.0088	-0.0266	-0.0565	-0.1036	-0.1635	-0.2323	-0.3182	-0.4071	-0.4928	-0.5897	-0.6758	-0.7457	-0.8203	-0.8726	-0.8996	-0.9272
163.00	-0.0090	-0.0274	-0.0582	-0.1068	-0.1686	-0.2396	-0.3282	-0.4198	-0.5082	-0.6082	-0.6971	-0.7691	-0.8460	-0.9000	-0.9279	-0.9563
167.00	-0.0092	-0.0279	-0.0593	-0.1088	-0.1718	-0.2441	-0.3344	-0.4278	-0.5178	-0.6197	-0.7102	-0.7837	-0.8620	-0.9170	-0.9454	-0.9744
171.00	-0.0093	-0.0283	-0.0601	-0.1103	-0.1742	-0.2475	-0.3390	-0.4336	-0.5249	-0.6281	-0.7199	-0.7944	-0.8738	-0.9296	-0.9583	-0.9877
174.00	-0.0094	-0.0285	-0.0605	-0.1110	-0.1754	-0.2492	-0.3413	-0.4366	-0.5285	-0.6325	-0.7249	-0.7999	-0.8798	-0.9360	-0.9650	-0.9945
176.00	-0.0094	-0.0286	-0.0607	-0.1114	-0.1759	-0.2499	-0.3423	-0.4379	-0.5301	-0.6344	-0.7271	-0.8023	-0.8825	-0.9389	-0.9679	-0.9976
178.00	-0.0094	-0.0286	-0.0608	-0.1116	-0.1762	-0.2504	-0.3430	-0.4387	-0.5311	-0.6356	-0.7284	-0.8038	-0.8841	-0.9406	-0.9697	-0.9994
179.00	-0.0094	-0.0287	-0.0608	-0.1116	-0.1763	-0.2505	-0.3431	-0.4389	-0.5313	-0.6358	-0.7288	-0.8041	-0.8845	-0.9410	-0.9701	-0.9998
180.00	-0.0094	-0.0287	-0.0608	-0.1116	-0.1763	-0.2505	-0.3432	-0.4390	-0.5314	-0.6359	-0.7289	-0.8043	-0.8847	-0.9412	-0.9703	-1.0000

XCTR=-0.00

EXACT ANSWERS

THETA \ RHO	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
0.	0.0080	0.0270	0.0593	0.1106	0.1756	0.2500	0.3430	0.4390	0.5314	0.6361	0.7290	0.8044	0.8847	0.9412	0.9703	1.0000
14.00	0.0078	0.0262	0.0576	0.1073	0.1704	0.2426	0.3328	0.4259	0.5157	0.6172	0.7073	0.7805	0.8585	0.9132	0.9415	0.9703
28.00	0.0071	0.0238	0.0524	0.0976	0.1551	0.2208	0.3029	0.3876	0.4692	0.5616	0.6437	0.7102	0.7812	0.8310	0.8567	0.8829
41.00	0.0060	0.0204	0.0448	0.0835	0.1325	0.1887	0.2589	0.3313	0.4011	0.4800	0.5502	0.6071	0.6677	0.7103	0.7323	0.7547
54.00	0.0047	0.0159	0.0349	0.0650	0.1032	0.1470	0.2016	0.2580	0.3124	0.3739	0.4285	0.4728	0.5200	0.5532	0.5703	0.5878
66.00	0.0033	0.0110	0.0241	0.0450	0.0714	0.1017	0.1395	0.1785	0.2162	0.2587	0.2965	0.3272	0.3599	0.3828	0.3947	0.4067
77.00	0.0018	0.0061	0.0133	0.0249	0.0395	0.0562	0.0772	0.0987	0.1195	0.1431	0.1640	0.1809	0.1990	0.2117	0.2183	0.2250
88.00	0.0003	0.0009	0.0021	0.0039	0.0061	0.0087	0.0120	0.0153	0.0185	0.0222	0.0254	0.0281	0.0309	0.0328	0.0339	0.0349
98.00	-0.0011	-0.0038	-0.0083	-0.0154	-0.0244	-0.0348	-0.0477	-0.0611	-0.0740	-0.0885	-0.1015	-0.1119	-0.1231	-0.1310	-0.1350	-0.1392
108.00	-0.0025	-0.0083	-0.0183	-0.0342	-0.0543	-0.0773	-0.1060	-0.1357	-0.1642	-0.1966	-0.2253	-0.2486	-0.2734	-0.2908	-0.2998	-0.3090
117.00	-0.0036	-0.0123	-0.0269	-0.0502	-0.0797	-0.1135	-0.1557	-0.1993	-0.2413	-0.2888	-0.3310	-0.3652	-0.4017	-0.4273	-0.4405	-0.4540
125.00	-0.0046	-0.0155	-0.0340	-0.0634	-0.1007	-0.1434	-0.1967	-0.2518	-0.3048	-0.3648	-0.4181	-0.4614	-0.5075	-0.5398	-0.5565	-0.5736
133.00	-0.0055	-0.0184	-0.0405	-0.0754	-0.1198	-0.1705	-0.2339	-0.2994	-0.3624	-0.4338	-0.4972	-0.5486	-0.6034	-0.6419	-0.6617	-0.6820
140.00	-0.0061	-0.0207	-0.0454	-0.0847	-0.1345	-0.1915	-0.2628	-0.3363	-0.4071	-0.4872	-0.5584	-0.6162	-0.6777	-0.7210	-0.7433	-0.7660
147.00	-0.0067	-0.0226	-0.0497	-0.0928	-0.1473	-0.2097	-0.2877	-0.3682	-0.4457	-0.5334	-0.6114	-0.6746	-0.7420	-0.7893	-0.8138	-0.8387
153.00	-0.0071	-0.0241	-0.0529	-0.0985	-0.1565	-0.2228	-0.3056	-0.3911	-0.4735	-0.5667	-0.6495	-0.7167	-0.7883	-0.8386	-0.8645	-0.8910
158.00	-0.0074	-0.0250	-0.0550	-0.1025	-0.1628	-0.2318	-0.3180	-0.4070	-0.4927	-0.5897	-0.6759	-0.7458	-0.8203	-0.8727	-0.8996	-0.9272
163.00	-0.0077	-0.0258	-0.0567	-0.1058	-0.1679	-0.2391	-0.3280	-0.4198	-0.5082	-0.6083	-0.6971	-0.7692	-0.8461	-0.9001	-0.9279	-0.9563
167.00	-0.0078	-0.0263	-0.0578	-0.1078	-0.1711	-0.2436	-0.3342	-0.4277	-0.5178	-0.6198	-0.7103	-0.7837	-0.8621	-0.9171	-0.9454	-0.9744
171.00	-0.0079	-0.0267	-0.0586	-0.1092	-0.1735	-0.2470	-0.3388	-0.4336	-0.5249	-0.6282	-0.7200	-0.7945	-0.8738	-0.9296	-0.9584	-0.9877
174.00	-0.0080	-0.0269	-0.0590	-0.1100	-0.1747	-0.2487	-0.3411	-0.4366	-0.5285	-0.6326	-0.7250	-0.8000	-0.8799	-0.9360	-0.9650	-0.9945
176.00	-0.0080	-0.0269	-0.0592	-0.1103	-0.1752	-0.2494	-0.3422	-0.4379	-0.5301	-0.6345	-0.7272	-0.8024	-0.8826	-0.9389	-0.9679	-0.9976
178.00	-0.0080	-0.0270	-0.0593	-0.1105	-0.1755	-0.2499	-0.3428	-0.4387	-0.5311	-0.6357	-0.7286	-0.8039	-0.8842	-0.9406	-0.9697	-0.9994
179.00	-0.0080	-0.0270	-0.0593	-0.1106	-0.1756	-0.2500	-0.3429	-0.4389	-0.5314	-0.6360	-0.7289	-0.8042	-0.8846	-0.9410	-0.9702	-0.9998
180.00	-0.0080	-0.0270	-0.0593	-0.1106	-0.1756	-0.2500	-0.3430	-0.4390	-0.5314	-0.6361	-0.7290	-0.8044	-0.8847	-0.9412	-0.9703	-1.0000

XCTR=-0.00

RESIDUALS- RHO

THETA	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
0.	0.0009	0.0012	0.0012	0.0008	0.0005	0.0004	0.0002	0.0000	0.0000	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	0.
14.00	0.0009	0.0012	0.0011	0.0008	0.0005	0.0004	0.0001	0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	0.
28.00	0.0008	0.0010	0.0010	0.0006	0.0004	0.0003	0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	0.
41.00	0.0006	0.0008	0.0008	0.0005	0.0003	0.0002	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	0.
54.00	0.0004	0.0006	0.0005	0.0003	0.0001	0.0000	-0.0001	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0000	-0.0000	0.
66.00	0.0002	0.0003	0.0002	0.0001	-0.0000	-0.0001	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0000	-0.0000	0.
77.00	-0.0000	0.0000	-0.0000	-0.0001	-0.0001	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	0.
88.00	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	0.
98.00	-0.0005	-0.0005	-0.0005	-0.0004	-0.0004	-0.0003	-0.0003	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	-0.
108.00	-0.0007	-0.0008	-0.0007	-0.0006	-0.0005	-0.0004	-0.0003	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	-0.
117.00	-0.0008	-0.0010	-0.0009	-0.0007	-0.0006	-0.0005	-0.0003	-0.0002	-0.0002	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	-0.0000	-0.
125.00	-0.0010	-0.0011	-0.0011	-0.0008	-0.0006	-0.0005	-0.0003	-0.0002	-0.0001	-0.0000	-0.0000	-0.0000	0.0000	0.0000	0.0000	-0.
133.00	-0.0011	-0.0013	-0.0012	-0.0009	-0.0006	-0.0005	-0.0003	-0.0002	-0.0001	-0.0000	-0.0000	0.0000	0.0000	0.0000	0.0000	-0.
140.00	-0.0012	-0.0014	-0.0013	-0.0009	-0.0007	-0.0005	-0.0003	-0.0001	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.
147.00	-0.0013	-0.0015	-0.0014	-0.0010	-0.0007	-0.0005	-0.0002	-0.0001	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.
153.00	-0.0013	-0.0015	-0.0014	-0.0010	-0.0007	-0.0005	-0.0002	-0.0001	-0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	-0.
158.00	-0.0014	-0.0016	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0001	-0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
163.00	-0.0014	-0.0016	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0001	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
167.00	-0.0014	-0.0016	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
171.00	-0.0014	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
174.00	-0.0014	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
176.00	-0.0014	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
178.00	-0.0014	-0.0017	-0.0015	-0.0011	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
179.00	-0.0014	-0.0017	-0.0015	-0.0011	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
180.00	-0.0014	-0.0017	-0.0015	-0.0011	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.

TEST CASE 2, THETAO=45 DEGREES.
(NO AXIAL BOUNDARY VALUES).

INPUT-								RHO								
THETA	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
45.00	0.01	0.02	0.04	0.08	0.12	0.18	0.24	0.31	0.38	0.45	0.52	0.57	0.63	0.67	0.69	0.71
56.00	1.12	1.68	2.18	2.68	3.13	3.52	3.91	4.25	4.53	4.81	5.03	5.20	5.37	5.48	5.54	0.56
66.00	0.81	1.22	1.59	1.95	2.28	2.56	2.85	3.09	3.29	3.50	3.66	3.78	3.90	3.99	4.03	0.41
76.00	0.48	0.73	0.94	1.16	1.35	1.52	1.69	1.84	1.96	2.08	2.18	2.25	2.32	2.37	2.40	0.24
86.00	0.14	0.21	0.27	0.33	0.39	0.44	0.49	0.53	0.57	0.60	0.63	0.65	0.67	0.68	0.69	0.07
95.00	-0.17	-0.26	-0.34	-0.42	-0.49	-0.55	-0.61	-0.66	-0.71	-0.75	-0.78	-0.81	-0.84	-0.85	-0.86	-0.09
103.00	-0.45	-0.67	-0.88	-1.08	-1.26	-1.42	-1.57	-1.71	-1.82	-1.93	-2.02	-2.09	-2.16	-2.20	-2.23	-0.22
111.00	-0.72	-1.08	-1.40	-1.72	-2.01	-2.26	-2.51	-2.72	-2.90	-3.08	-3.23	-3.33	-3.44	-3.51	-3.55	-0.36
119.00	-0.97	-1.45	-1.89	-2.33	-2.71	-3.05	-3.39	-3.68	-3.93	-4.17	-4.36	-4.51	-4.65	-4.75	-4.80	-0.48
126.00	-1.18	-1.76	-2.29	-2.82	-3.29	-3.70	-4.11	-4.47	-4.76	-5.05	-5.29	-5.47	-5.64	-5.76	-5.82	-0.59
133.00	-1.36	-2.05	-2.66	-3.27	-3.82	-4.30	-4.77	-5.18	-5.52	-5.87	-6.14	-6.34	-6.55	-6.68	-6.75	-0.68
139.00	-1.51	-2.26	-2.94	-3.62	-4.23	-4.75	-5.28	-5.74	-6.11	-6.49	-6.79	-7.02	-7.25	-7.40	-7.47	-0.75
145.00	-1.64	-2.46	-3.19	-3.93	-4.59	-5.16	-5.73	-6.23	-6.64	-7.04	-7.37	-7.62	-7.86	-8.03	-8.11	-0.82
150.00	-1.73	-2.60	-3.38	-4.16	-4.85	-5.46	-6.06	-6.58	-7.01	-7.45	-7.79	-8.05	-8.31	-8.49	-8.57	-0.87
155.00	-1.81	-2.72	-3.53	-4.35	-5.08	-5.71	-6.34	-6.89	-7.34	-7.79	-8.16	-8.43	-8.70	-8.88	-8.97	-0.91
160.00	-1.88	-2.82	-3.66	-4.51	-5.26	-5.92	-6.58	-7.14	-7.61	-8.08	-8.46	-8.74	-9.02	-9.21	-9.30	-0.94
164.00	-1.92	-2.88	-3.75	-4.61	-5.38	-6.06	-6.73	-7.31	-7.79	-8.27	-8.65	-8.94	-9.23	-9.42	-9.52	-0.96
168.00	-1.96	-2.93	-3.81	-4.70	-5.48	-6.16	-6.85	-7.43	-7.92	-8.41	-8.80	-9.10	-9.39	-9.59	-9.68	-0.98
171.00	-1.98	-2.96	-3.85	-4.74	-5.53	-6.22	-6.91	-7.51	-8.00	-8.49	-8.89	-9.19	-9.48	-9.68	-9.78	-0.99
174.00	-1.99	-2.98	-3.88	-4.77	-5.57	-6.27	-6.96	-7.56	-8.06	-8.55	-8.95	-9.25	-9.55	-9.75	-9.85	-0.99
176.00	-2.00	-2.99	-3.89	-4.79	-5.59	-6.28	-6.98	-7.58	-8.08	-8.58	-8.98	-9.28	-9.58	-9.78	-9.88	-1.00
177.00	-2.00	-3.00	-3.89	-4.79	-5.59	-6.29	-6.99	-7.59	-8.09	-8.59	-8.99	-9.29	-9.59	-9.79	-9.89	-1.00
178.00	-2.00	-3.00	-3.90	-4.80	-5.60	-6.30	-7.00	-7.60	-8.10	-8.59	-8.99	-9.29	-9.59	-9.79	-9.89	-1.00
179.00	-2.00	-3.00	-3.90	-4.80	-5.60	-6.30	-7.00	-7.60	-8.10	-8.60	-9.00	-9.30	-9.60	-9.80	-9.90	-1.00
180.00	-2.00	-3.00	-3.90	-4.80	-5.60	-6.30	-7.00	-7.60	-8.10	-8.60	-9.00	-9.30	-9.60	-9.80	-9.90	-1.00

OUTPUT- RHO

THETA	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
45.00	0.0057	0.0191	0.0419	0.0782	0.1242	0.1768	0.2425	0.3104	0.3758	0.4498	0.5155	0.5688	0.6256	0.6655	0.6861	0.7071
56.00	0.0044	0.0151	0.0332	0.0618	0.0981	0.1397	0.1917	0.2453	0.2970	0.3555	0.4075	0.4497	0.4947	0.5263	0.5[illegible]26	0.5592
66.00	0.0031	0.0109	0.0241	0.0449	0.0713	0.1016	0.1393	0.1784	0.2160	0.2586	0.2964	0.3271	0.3598	0.3828	0.3946	0.4067
76.00	0.0016	0.0063	0.0142	0.0265	0.0423	0.0603	0.0828	0.1060	0.1284	0.1537	0.1763	0.1945	0.2140	0.2277	0.2347	0.2419
86.00	0.0001	0.0015	0.0038	0.0074	0.0119	0.0171	0.0236	0.0304	0.0369	0.0442	0.0507	0.0560	0.0617	0.0656	0.0677	0.0698
95.00	-0.0013	-0.0029	-0.0057	-0.0101	-0.0157	-0.0221	-0.0302	-0.0385	-0.0465	-0.0556	-0.0637	-0.0702	-0.0772	-0.0821	-0.0846	-0.0872
103.00	-0.0025	-0.0068	-0.0140	-0.0254	-0.0399	-0.0566	-0.0774	-0.0989	-0.1197	-0.1432	-0.1640	-0.1810	-0.1990	-0.2117	-0.2183	-0.2250
111.00	-0.0037	-0.0106	-0.0221	-0.0403	-0.0634	-0.0900	-0.1232	-0.1575	-0.1906	-0.2280	-0.2613	-0.2883	-0.3171	-0.3373	-0.3477	-0.3584
119.00	-0.0048	-0.0141	-0.0297	-0.0543	-0.0857	-0.1216	-0.1665	-0.2130	-0.2578	-0.3084	-0.3535	-0.3900	-0.4289	-0.4563	-0.4704	-0.4848
126.00	-0.0058	-0.0170	-0.0360	-0.0658	-0.1038	-0.1474	-0.2019	-0.2582	-0.3125	-0.3739	-0.4285	-0.4728	-0.5200	-0.5532	-0.5703	-0.5878
133.00	-0.0066	-0.0197	-0.0416	-0.0763	-0.1204	-0.1710	-0.2342	-0.2995	-0.3625	-0.4338	-0.4972	-0.5486	-0.6034	-0.6419	-0.6617	-0.6820
139.00	-0.0073	-0.0218	-0.0460	-0.0844	-0.1332	-0.1892	-0.2591	-0.3314	-0.4011	-0.4800	-0.5501	-0.6070	-0.6677	-0.7103	-0.7323	-0.7547
145.00	-0.0078	-0.0236	-0.0499	-0.0915	-0.1445	-0.2053	-0.2812	-0.3597	-0.4354	-0.5210	-0.5971	-0.6588	-0.7247	-0.7710	-0.7948	-0.8192
150.00	-0.0082	-0.0249	-0.0528	-0.0967	-0.1527	-0.2170	-0.2972	-0.3802	-0.4602	-0.5508	-0.6312	-0.6965	-0.7662	-0.8151	-0.8403	-0.8660
155.00	-0.0086	-0.0260	-0.0552	-0.1012	-0.1598	-0.2271	-0.3110	-0.3979	-0.4816	-0.5764	-0.6606	-0.7289	-0.8018	-0.8530	-0.8794	-0.9063
160.00	-0.0089	-0.0270	-0.0572	-0.1049	-0.1657	-0.2354	-0.3225	-0.4125	-0.4994	-0.5976	-0.6849	-0.7558	-0.8313	-0.8844	-0.9118	-0.9397
164.00	-0.0091	-0.0276	-0.0585	-0.1073	-0.1695	-0.2408	-0.3299	-0.4220	-0.5108	-0.6113	-0.7006	-0.7731	-0.8504	-0.9047	-0.9327	-0.9613
168.00	-0.0093	-0.0281	-0.0595	-0.1092	-0.1724	-0.2450	-0.3357	-0.4294	-0.5198	-0.6220	-0.7129	-0.7867	-0.8653	-0.9206	-0.9491	-0.9781
171.00	-0.0093	-0.0283	-0.0601	-0.1103	-0.1741	-0.2474	-0.3389	-0.4336	-0.5249	-0.5281	-0.7199	-0.7944	-0.8738	-0.9296	-0.9583	-0.9877
174.00	-0.0094	-0.0285	-0.0605	-0.1110	-0.1753	-0.2491	-0.3413	-0.4366	-0.5285	-0.6324	-0.7249	-0.7999	-0.8798	-0.9360	-0.9650	-0.9945
176.00	-0.0094	-0.0286	-0.0607	-0.1113	-0.1759	-0.2499	-0.3423	-0.4379	-0.5301	-0.6344	-0.7271	-0.8023	-0.8825	-0.9389	-0.9679	-0.9976
177.00	-0.0094	-0.0286	-0.0608	-0.1115	-0.1760	-0.2502	-0.3427	-0.4384	-0.5307	-0.6351	-0.7279	-0.8032	-0.8834	-0.9399	-0.9689	-0.9986
178.00	-0.0094	-0.0287	-0.0608	-0.1116	-0.1762	-0.2503	-0.3429	-0.4387	-0.5311	-0.6355	-0.7284	-0.8038	-0.8841	-0.9406	-0.9697	-0.9994
179.00	-0.0094	-0.0287	-0.0608	-0.1116	-0.1763	-0.2505	-0.3431	-0.4389	-0.5313	-0.6358	-0.7288	-0.8041	-0.8845	-0.9410	-0.9701	-0.9998
180.00	-0.0095	-0.0287	-0.0608	-0.1116	-0.1763	-0.2505	-0.3432	-0.4390	-0.5314	-0.6359	-0.7289	-0.8043	-0.8847	-0.9411	-0.9703	-1.0000

XCTR= 0.

EXACT ANSWERS RHO

THETA	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
45.00	0.0057	0.0191	0.0419	0.0782	0.1242	0.1768	0.2425	0.3104	0.3758	0.4498	0.5155	0.5688	0.6256	0.6655	0.6861	0.7071
56.00	0.0045	0.0151	0.0332	0.0618	0.0982	0.1398	0.1918	0.2455	0.2972	0.3557	0.4077	0.4498	0.4947	0.5263	0.5426	0.5592
66.00	0.0033	0.0110	0.0241	0.0450	0.0714	0.1017	0.1395	0.1785	0.2162	0.2587	0.2965	0.3272	0.3599	0.3828	0.3947	0.4067
76.00	0.0019	0.0065	0.0144	0.0268	0.0425	0.0605	0.0830	0.1062	0.1286	0.1539	0.1764	0.1946	0.2140	0.2277	0.2347	0.2419
86.00	0.0006	0.0019	0.0041	0.0077	0.0123	0.0174	0.0239	0.0306	0.0371	0.0444	0.0509	0.0561	0.0617	0.0657	0.0677	0.0698
95.00	-0.0007	-0.0024	-0.0052	-0.0096	-0.0153	-0.0218	-0.0299	-0.0383	-0.0463	-0.0554	-0.0635	-0.0701	-0.0771	-0.0820	-0.0846	-0.0872
103.00	-0.0018	-0.0061	-0.0133	-0.0249	-0.0395	-0.0562	-0.0772	-0.0987	-0.1195	-0.1431	-0.1640	-0.1809	-0.1990	-0.2117	-0.2183	-0.2250
111.00	-0.0029	-0.0097	-0.0213	-0.0396	-0.0629	-0.0896	-0.1229	-0.1573	-0.1905	-0.2279	-0.2613	-0.2883	-0.3171	-0.3373	-0.3477	-0.3584
119.00	-0.0039	-0.0131	-0.0288	-0.0536	-0.0851	-0.1212	-0.1663	-0.2128	-0.2576	-0.3084	-0.3534	-0.3900	-0.4289	-0.4563	-0.4704	-0.4848
126.00	-0.0047	-0.0159	-0.0349	-0.0650	-0.1032	-0.1470	-0.2016	-0.2580	-0.3124	-0.3739	-0.4285	-0.4728	-0.5200	-0.5532	-0.5703	-0.5878
133.00	-0.0055	-0.0184	-0.0405	-0.0754	-0.1198	-0.1705	-0.2339	-0.2994	-0.3624	-0.4338	-0.4972	-0.5486	-0.6034	-0.6419	-0.6617	-0.6820
139.00	-0.0060	-0.0204	-0.0448	-0.0835	-0.1325	-0.1887	-0.2589	-0.3313	-0.4011	-0.4800	-0.5502	-0.6071	-0.6677	-0.7103	-0.7323	-0.7547
145.00	-0.0066	-0.0221	-0.0486	-0.0906	-0.1439	-0.2048	-0.2810	-0.3596	-0.4353	-0.5210	-0.5972	-0.6589	-0.7247	-0.7710	-0.7948	-0.8192
150.00	-0.0069	-0.0234	-0.0514	-0.0958	-0.1521	-0.2165	-0.2970	-0.3802	-0.4602	-0.5508	-0.6313	-0.6966	-0.7662	-0.8151	-0.8403	-0.8660
155.00	-0.0073	-0.0245	-0.0538	-0.1002	-0.1592	-0.2266	-0.3109	-0.3978	-0.4816	-0.5765	-0.6607	-0.7290	-0.8018	-0.8530	-0.8794	-0.9063
160.00	-0.0075	-0.0254	-0.0557	-0.1039	-0.1650	-0.2350	-0.3223	-0.4125	-0.4994	-0.5977	-0.6850	-0.7558	-0.8314	-0.8844	-0.9118	-0.9397
164.00	-0.0077	-0.0260	-0.0570	-0.1063	-0.1688	-0.2404	-0.3297	-0.4220	-0.5109	-0.6114	-0.7008	-0.7732	-0.8505	-0.9047	-0.9327	-0.9613
168.00	-0.0078	-0.0264	-0.0580	-0.1082	-0.1718	-0.2446	-0.3355	-0.4294	-0.5198	-0.6222	-0.7131	-0.7868	-0.8654	-0.9206	-0.9491	-0.9781
171.00	-0.0079	-0.0267	-0.0586	-0.1092	-0.1735	-0.2470	-0.3388	-0.4336	-0.5249	-0.6282	-0.7200	-0.7945	-0.8738	-0.9296	-0.9584	-0.9877
174.00	-0.0080	-0.0269	-0.0590	-0.1100	-0.1747	-0.2487	-0.3411	-0.4366	-0.5285	-0.6326	-0.7250	-0.8000	-0.8799	-0.9360	-0.9650	-0.9945
176.00	-0.0080	-0.0269	-0.0592	-0.1103	-0.1752	-0.2494	-0.3422	-0.4379	-0.5301	-0.6345	-0.7272	-0.8024	-0.8826	-0.9389	-0.9679	-0.9976
177.00	-0.0080	-0.0270	-0.0592	-0.1104	-0.1754	-0.2497	-0.3425	-0.4384	-0.5307	-0.6352	-0.7280	-0.8033	-0.8835	-0.9399	-0.9690	-0.9986
178.00	-0.0080	-0.0270	-0.0593	-0.1105	-0.1755	-0.2499	-0.3428	-0.4387	-0.5311	-0.6357	-0.7286	-0.8039	-0.8842	-0.9406	-0.9697	-0.9994
179.00	-0.0080	-0.0270	-0.0593	-0.1106	-0.1756	-0.2500	-0.3429	-0.4389	-0.5314	-0.6360	-0.7289	-0.8042	-0.8846	-0.9410	-0.9702	-0.9998
180.00	-0.0080	-0.0270	-0.0593	-0.1106	-0.1756	-0.2500	-0.3430	-0.4390	-0.5314	-0.6361	-0.7290	-0.8044	-0.8847	-0.9412	-0.9703	-1.0000

XCTR= 0.

RESIDUALS- RHO

THETA	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
45.00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
56.00	-0.0001	0.0000	0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	0.
66.00	-0.0002	-0.0001	-0.0000	-0.0001	-0.0001	-0.0001	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	0.
76.00	-0.0003	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	0.
86.00	-0.0004	-0.0004	-0.0004	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	0.
95.00	-0.0006	-0.0006	-0.0005	-0.0005	-0.0004	-0.0003	-0.0003	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.
103.00	-0.0007	-0.0007	-0.0007	-0.0005	-0.0004	-0.0004	-0.0003	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	-0.0000	-0.
111.00	-0.0008	-0.0009	-0.0008	-0.0006	-0.0005	-0.0004	-0.0002	-0.0002	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	0.0000	0.0000	-0.
119.00	-0.0010	-0.0011	-0.0010	-0.0007	-0.0005	-0.0004	-0.0003	-0.0002	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.
126.00	-0.0011	-0.0012	-0.0011	-0.0008	-0.0006	-0.0004	-0.0002	-0.0001	-0.0001	-0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.
133.00	-0.0011	-0.0013	-0.0012	-0.0009	-0.0006	-0.0005	-0.0002	-0.0001	-0.0001	-0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.
139.00	-0.0012	-0.0014	-0.0013	-0.0009	-0.0006	-0.0005	-0.0002	-0.0001	-0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.
145.00	-0.0013	-0.0015	-0.0013	-0.0009	-0.0006	-0.0005	-0.0002	-0.0001	-0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	-0.
150.00	-0.0013	-0.0015	-0.0014	-0.0010	-0.0006	-0.0005	-0.0002	-0.0001	-0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
155.00	-0.0014	-0.0016	-0.0014	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
160.00	-0.0014	-0.0016	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
164.00	-0.0014	-0.0016	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
168.00	-0.0014	-0.0016	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
171.00	-0.0014	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
174.00	-0.0014	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
176.00	-0.0014	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	-0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
177.00	-0.0014	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
178.00	-0.0014	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
179.00	-0.0015	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
180.00	-0.0015	-0.0017	-0.0015	-0.0010	-0.0007	-0.0005	-0.0002	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.

TEST CASE 3, THETAO=60 DEGREES.
(WITH AXIAL BOUNDARY VALUES).

AXIAL VALUES-

R	V
0.	0.
0.200	-0.008
0.300	-0.027
0.390	-0.059
0.480	-0.111
0.560	-0.176
0.630	-0.250
0.700	-0.343
0.760	-0.439
0.810	-0.531
0.860	-0.636
0.900	-0.729
0.930	-0.804
0.960	-0.885
0.980	-0.941
0.990	-0.970
1.000	-1.000

INPUT-	RHO															
THETA	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
60.00	0.00	0.01	0.03	0.06	0.09	0.13	0.17	0.22	0.27	0.32	0.36	0.40	0.44	0.47	0.49	0.50
70.00	0.68	1.03	1.33	1.64	1.92	2.15	2.39	2.60	2.77	2.94	3.08	3.18	3.28	3.35	3.39	0.34
79.00	0.38	0.57	0.74	0.92	1.07	1.20	1.34	1.45	1.55	1.64	1.72	1.77	1.83	1.87	1.89	0.19
88.00	0.07	0.10	0.14	0.17	0.20	0.22	0.24	0.27	0.28	0.30	0.31	0.32	0.34	0.34	0.35	0.03
96.00	-0.21	-0.31	-0.41	-0.50	-0.59	-0.66	-0.73	-0.79	-0.85	-0.90	-0.94	-0.97	-1.00	-1.02	-1.03	-0.10
104.00	-0.48	-0.73	-0.94	-1.16	-1.35	-1.52	-1.69	-1.84	-1.96	-2.08	-2.18	-2.25	-2.32	-2.37	-2.40	-0.24
112.00	-0.75	-1.12	-1.46	-1.80	-2.10	-2.36	-2.62	-2.85	-3.03	-3.22	-3.37	-3.48	-3.60	-3.67	-3.71	-0.37
119.00	-0.97	-1.45	-1.89	-2.33	-2.71	-3.05	-3.39	-3.68	-3.93	-4.17	-4.36	-4.51	-4.65	-4.75	-4.80	-0.48
126.00	-1.18	-1.76	-2.29	-2.82	-3.29	-3.70	-4.11	-4.47	-4.76	-5.05	-5.29	-5.47	-5.64	-5.76	-5.82	-0.59
132.00	-1.34	-2.01	-2.61	-3.21	-3.75	-4.22	-4.68	-5.09	-5.42	-5.75	-6.02	-6.22	-6.42	-6.56	-6.62	-0.67
138.00	-1.49	-2.23	-2.90	-3.57	-4.16	-4.68	-5.20	-5.65	-6.02	-6.39	-6.69	-6.91	-7.13	-7.28	-7.36	-0.74
144.00	-1.62	-2.43	-3.16	-3.88	-4.53	-5.10	-5.66	-6.15	-6.55	-6.96	-7.28	-7.52	-7.77	-7.93	-8.01	-0.81
149.00	-1.71	-2.57	-3.34	-4.11	-4.80	-5.40	-6.00	-6.51	-6.94	-7.37	-7.71	-7.97	-8.23	-8.40	-8.49	-0.86
154.00	-1.80	-2.70	-3.51	-4.31	-5.03	-5.66	-6.29	-6.83	-7.28	-7.73	-8.09	-8.36	-8.63	-8.81	-8.90	-0.90
158.00	-1.85	-2.78	-3.62	-4.45	-5.19	-5.84	-6.49	-7.05	-7.51	-7.97	-8.34	-8.62	-8.90	-9.09	-9.18	-0.93
162.00	-1.90	-2.85	-3.71	-4.57	-5.33	-5.99	-6.66	-7.23	-7.70	-8.18	-8.56	-8.84	-9.13	-9.32	-9.42	-0.95
166.00	-1.94	-2.91	-3.78	-4.66	-5.43	-6.11	-6.79	-7.37	-7.86	-8.34	-8.73	-9.02	-9.31	-9.51	-9.61	-0.97
169.00	-1.96	-2.94	-3.83	-4.71	-5.50	-6.18	-6.87	-7.46	-7.95	-8.44	-8.83	-9.13	-9.42	-9.62	-9.72	-0.98
172.00	-1.98	-2.97	-3.86	-4.75	-5.55	-6.24	-6.93	-7.53	-8.02	-8.52	-8.91	-9.21	-9.51	-9.70	-9.80	-0.99
174.00	-1.99	-2.98	-3.88	-4.77	-5.57	-6.27	-6.96	-7.56	-8.06	-8.55	-8.95	-9.25	-9.55	-9.75	-9.85	-0.99
176.00	-2.00	-2.99	-3.89	-4.79	-5.59	-6.28	-6.98	-7.58	-8.08	-8.58	-8.98	-9.28	-9.58	-9.78	-9.88	-1.00
177.00	-2.00	-3.00	-3.89	-4.79	-5.59	-6.29	-6.99	-7.59	-8.09	-8.59	-8.99	-9.29	-9.59	-9.79	-9.89	-1.00
178.00	-2.00	-3.00	-3.90	-4.80	-5.60	-6.30	-7.00	-7.60	-8.10	-8.59	-8.99	-9.29	-9.59	-9.79	-9.89	-1.00
179.00	-2.00	-3.00	-3.90	-4.80	-5.60	-6.30	-7.00	-7.60	-8.10	-8.60	-9.00	-9.30	-9.60	-9.80	-9.90	-1.00
180.00	-0.01	-0.03	-0.06	-0.11	-0.18	-0.25	-0.34	-0.44	-0.53	-0.64	-0.73	-0.80	-0.88	-0.94	-0.97	-1.00

OUTPUT- RHO

THETA	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
60.00	0.0040	0.0135	0.0297	0.0553	0.0878	0.1250	0.1715	0.2195	0.2657	0.3180	0.3645	0.4022	0.4424	0.4706	0.4851	0.5000
70.00	0.0026	0.0091	0.0202	0.0377	0.0600	0.0854	0.1172	0.1500	0.1816	0.2174	0.2492	0.2750	0.3025	0.3219	0.3318	0.3420
79.00	0.0013	0.0049	0.0111	0.0209	0.0333	0.0475	0.0653	0.0836	0.1013	0.1212	0.1390	0.1534	0.1688	0.1796	0.1851	0.1908
88.00	-0.0001	0.0006	0.0017	0.0036	0.0059	0.0085	0.0118	0.0151	0.0184	0.0221	0.0253	0.0280	0.0308	0.0328	0.0338	0.0349
96.00	-0.0013	-0.0033	-0.0067	-0.0119	-0.0187	-0.0264	-0.0360	-0.0460	-0.0557	-0.0666	-0.0763	-0.0841	-0.0925	-0.0984	-0.1014	-0.1045
104.00	-0.0025	-0.0072	-0.0150	-0.0272	-0.0428	-0.0608	-0.0832	-0.1063	-0.1287	-0.1539	-0.1764	-0.1946	-0.2140	-0.2277	-0.2347	-0.2419
112.00	-0.0037	-0.0109	-0.0230	-0.0420	-0.0662	-0.0940	-0.1287	-0.1646	-0.1992	-0.2383	-0.2731	-0.3013	-0.3314	-0.3526	-0.3635	-0.3746
119.00	-0.0047	-0.0140	-0.0296	-0.0542	-0.0856	-0.1216	-0.1665	-0.2129	-0.2577	-0.3084	-0.3534	-0.3900	-0.4289	-0.4563	-0.4704	-0.4848
126.00	-0.0056	-0.0169	-0.0358	-0.0657	-0.1037	-0.1473	-0.2018	-0.2581	-0.3124	-0.3739	-0.4285	-0.4728	-0.5200	-0.5532	-0.5703	-0.5878
132.00	-0.0063	-0.0192	-0.0407	-0.0747	-0.1180	-0.1677	-0.2297	-0.2938	-0.3556	-0.4256	-0.4877	-0.5382	-0.5920	-0.6298	-0.6492	-0.6691
138.00	-0.0069	-0.0212	-0.0452	-0.0829	-0.1310	-0.1862	-0.2551	-0.3263	-0.3949	-0.4726	-0.5417	-0.5977	-0.6574	-0.6994	-0.7211	-0.7431
144.00	-0.0075	-0.0231	-0.0491	-0.0903	-0.1426	-0.2027	-0.2776	-0.3552	-0.4299	-0.5145	-0.5897	-0.6507	-0.7157	-0.7614	-0.7850	-0.8090
149.00	-0.0079	-0.0244	-0.0520	-0.0956	-0.1511	-0.2147	-0.2941	-0.3763	-0.4555	-0.5451	-0.6248	-0.6894	-0.7583	-0.8067	-0.8317	-0.8572
154.00	-0.0082	-0.0255	-0.0545	-0.1002	-0.1584	-0.2251	-0.3084	-0.3945	-0.4776	-0.5716	-0.6551	-0.7229	-0.7951	-0.8459	-0.8721	-0.8988
158.00	-0.0085	-0.0263	-0.0562	-0.1033	-0.1633	-0.2322	-0.3181	-0.4070	-0.4927	-0.5896	-0.6758	-0.7457	-0.8202	-0.8726	-0.8996	-0.9272
162.00	-0.0086	-0.0269	-0.0576	-0.1060	-0.1675	-0.2381	-0.3263	-0.4174	-0.5054	-0.6048	-0.6932	-0.7649	-0.8414	-0.8951	-0.9228	-0.9511
166.00	-0.0088	-0.0274	-0.0587	-0.1081	-0.1709	-0.2429	-0.3329	-0.4259	-0.5156	-0.6170	-0.7072	-0.7804	-0.8584	-0.9132	-0.9415	-0.9703
169.00	-0.0088	-0.0277	-0.0593	-0.1093	-0.1728	-0.2458	-0.3368	-0.4309	-0.5216	-0.6242	-0.7155	-0.7895	-0.8684	-0.9239	-0.9524	-0.9816
172.00	-0.0088	-0.0278	-0.0598	-0.1102	-0.1743	-0.2479	-0.3397	-0.4346	-0.5262	-0.6297	-0.7218	-0.7964	-0.8760	-0.9320	-0.9608	-0.9903
174.00	-0.0088	-0.0279	-0.0600	-0.1106	-0.1750	-0.2489	-0.3412	-0.4365	-0.5284	-0.6324	-0.7249	-0.7998	-0.8798	-0.9360	-0.9650	-0.9945
176.00	-0.0087	-0.0279	-0.0600	-0.1109	-0.1755	-0.2497	-0.3422	-0.4378	-0.5301	-0.6344	-0.7271	-0.8023	-0.8825	-0.9389	-0.9679	-0.9976
177.00	-0.0087	-0.0278	-0.0600	-0.1110	-0.1757	-0.2499	-0.3426	-0.4383	-0.5306	-0.6351	-0.7279	-0.8032	-0.8835	-0.9399	-0.9689	-0.9986
178.00	-0.0086	-0.0277	-0.0600	-0.1110	-0.1758	-0.2501	-0.3428	-0.4387	-0.5310	-0.6356	-0.7284	-0.8038	-0.8841	-0.9406	-0.9697	-0.9994
179.00	-0.0084	-0.0276	-0.0598	-0.1109	-0.1758	-0.2502	-0.3430	-0.4389	-0.5313	-0.6359	-0.7288	-0.8042	-0.8846	-0.9410	-0.9701	-0.9998
180.00	-0.0080	-0.0270	-0.0593	-0.1106	-0.1756	-0.2500	-0.3430	-0.4390	-0.5314	-0.6361	-0.7290	-0.8044	-0.8847	-0.9412	-0.9703	-1.0000

XCTR= 0.

EXACT ANSWERS RHO

THETA	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
60.00	0.0040	0.0135	0.0297	0.0553	0.0878	0.1250	0.1715	0.2195	0.2657	0.3180	0.3645	0.4022	0.4424	0.4706	0.4851	0.5000
70.00	0.0027	0.0092	0.0203	0.0378	0.0601	0.0855	0.1173	0.1501	0.1818	0.2175	0.2493	0.2751	0.3026	0.3219	0.3319	0.3420
79.00	0.0015	0.0052	0.0113	0.0211	0.0335	0.0477	0.0654	0.0838	0.1014	0.1214	0.1391	0.1535	0.1688	0.1796	0.1851	0.1908
88.00	0.0003	0.0009	0.0021	0.0039	0.0061	0.0087	0.0120	0.0153	0.0185	0.0222	0.0254	0.0281	0.0309	0.0328	0.0339	0.0349
96.00	-0.0008	-0.0028	-0.0062	-0.0116	-0.0184	-0.0261	-0.0359	-0.0459	-0.0556	-0.0665	-0.0762	-0.0841	-0.0925	-0.0984	-0.1014	-0.1045
104.00	-0.0019	-0.0065	-0.0144	-0.0268	-0.0425	-0.0605	-0.0830	-0.1062	-0.1286	-0.1539	-0.1764	-0.1946	-0.2140	-0.2277	-0.2347	-0.2419
112.00	-0.0030	-0.0101	-0.0222	-0.0414	-0.0658	-0.0937	-0.1285	-0.1644	-0.1991	-0.2383	-0.2731	-0.3013	-0.3314	-0.3526	-0.3635	-0.3746
119.00	-0.0039	-0.0131	-0.0288	-0.0536	-0.0851	-0.1212	-0.1663	-0.2128	-0.2576	-0.3084	-0.3534	-0.3900	-0.4289	-0.4563	-0.4704	-0.4848
126.00	-0.0047	-0.0159	-0.0349	-0.0650	-0.1032	-0.1470	-0.2016	-0.2580	-0.3124	-0.3739	-0.4285	-0.4728	-0.5200	-0.5532	-0.5703	-0.5878
132.00	-0.0054	-0.0181	-0.0397	-0.0740	-0.1175	-0.1673	-0.2295	-0.2937	-0.3556	-0.4256	-0.4878	-0.5382	-0.5920	-0.6298	-0.6493	-0.6691
138.00	-0.0059	-0.0201	-0.0441	-0.0822	-0.1305	-0.1858	-0.2549	-0.3262	-0.3949	-0.4727	-0.5418	-0.5978	-0.6575	-0.6994	-0.7211	-0.7431
144.00	-0.0065	-0.0218	-0.0480	-0.0895	-0.1421	-0.2023	-0.2775	-0.3551	-0.4299	-0.5146	-0.5898	-0.6507	-0.7158	-0.7614	-0.7850	-0.8090
149.00	-0.0069	-0.0231	-0.0508	-0.0948	-0.1505	-0.2143	-0.2940	-0.3763	-0.4555	-0.5452	-0.6249	-0.6895	-0.7584	-0.8068	-0.8317	-0.8572
154.00	-0.0072	-0.0243	-0.0533	-0.0994	-0.1578	-0.2247	-0.3083	-0.3945	-0.4777	-0.5717	-0.6552	-0.7230	-0.7952	-0.8459	-0.8721	-0.8988
158.00	-0.0074	-0.0250	-0.0550	-0.1025	-0.1628	-0.2318	-0.3180	-0.4070	-0.4927	-0.5897	-0.6759	-0.7458	-0.8203	-0.8727	-0.8996	-0.9272
162.00	-0.0076	-0.0257	-0.0564	-0.1052	-0.1670	-0.2378	-0.3262	-0.4175	-0.5054	-0.6049	-0.6933	-0.7650	-0.8414	-0.8951	-0.9228	-0.9511
166.00	-0.0078	-0.0262	-0.0576	-0.1073	-0.1704	-0.2426	-0.3328	-0.4259	-0.5157	-0.6172	-0.7073	-0.7805	-0.8585	-0.9132	-0.9415	-0.9703
169.00	-0.0079	-0.0265	-0.0582	-0.1086	-0.1724	-0.2455	-0.3367	-0.4309	-0.5217	-0.6244	-0.7156	-0.7896	-0.8685	-0.9239	-0.9525	-0.9816
172.00	-0.0079	-0.0267	-0.0587	-0.1095	-0.1739	-0.2476	-0.3397	-0.4347	-0.5263	-0.6299	-0.7219	-0.7965	-0.8761	-0.9320	-0.9609	-0.9903
174.00	-0.0080	-0.0269	-0.0590	-0.1100	-0.1747	-0.2487	-0.3411	-0.4366	-0.5285	-0.6326	-0.7250	-0.8000	-0.8799	-0.9360	-0.9650	-0.9945
176.00	-0.0080	-0.0269	-0.0592	-0.1103	-0.1752	-0.2494	-0.3422	-0.4379	-0.5301	-0.6345	-0.7272	-0.8024	-0.8826	-0.9389	-0.9679	-0.9976
177.00	-0.0080	-0.0270	-0.0592	-0.1104	-0.1754	-0.2497	-0.3425	-0.4384	-0.5307	-0.6352	-0.7280	-0.8033	-0.8835	-0.9399	-0.9690	-0.9986
178.00	-0.0080	-0.0270	-0.0593	-0.1105	-0.1755	-0.2499	-0.3428	-0.4387	-0.5311	-0.6357	-0.7286	-0.8039	-0.8842	-0.9406	-0.9697	-0.9994
179.00	-0.0080	-0.0270	-0.0593	-0.1106	-0.1756	-0.2500	-0.3429	-0.4389	-0.5314	-0.6360	-0.7289	-0.8042	-0.8846	-0.9410	-0.9702	-0.9998
180.00	-0.0080	-0.0270	-0.0593	-0.1106	-0.1756	-0.2500	-0.3430	-0.4390	-0.5314	-0.6361	-0.7290	-0.8044	-0.8847	-0.9412	-0.9703	-1.0000

XCTR= 0.

RESIDUALS-

THETA \ RHO	0.200	0.300	0.390	0.480	0.560	0.630	0.700	0.760	0.810	0.860	0.900	0.930	0.960	0.980	0.990	1.000
60.00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
70.00	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	0.
79.00	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	0.
88.00	-0.0004	-0.0004	-0.0003	-0.0003	-0.0003	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	0.
96.00	-0.0005	-0.0005	-0.0005	-0.0004	-0.0003	-0.0003	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	-0.0000	-0.
104.00	-0.0006	-0.0007	-0.0006	-0.0005	-0.0004	-0.0003	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.
112.00	-0.0007	-0.0008	-0.0008	-0.0006	-0.0004	-0.0003	-0.0002	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.
119.00	-0.0008	-0.0009	-0.0009	-0.0006	-0.0005	-0.0004	-0.0002	-0.0001	-0.0001	-0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.
126.00	-0.0009	-0.0010	-0.0010	-0.0007	-0.0005	-0.0004	-0.0002	-0.0001	-0.0001	-0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.
132.00	-0.0009	-0.0011	-0.0010	-0.0007	-0.0005	-0.0004	-0.0002	-0.0001	-0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	-0.
138.00	-0.0010	-0.0012	-0.0011	-0.0008	-0.0005	-0.0004	-0.0002	-0.0000	-0.0000	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	-0.
144.00	-0.0010	-0.0012	-0.0011	-0.0008	-0.0005	-0.0004	-0.0001	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	-0.
149.00	-0.0010	-0.0013	-0.0012	-0.0008	-0.0005	-0.0004	-0.0001	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
154.00	-0.0010	-0.0013	-0.0012	-0.0008	-0.0005	-0.0004	-0.0001	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
158.00	-0.0010	-0.0013	-0.0012	-0.0008	-0.0005	-0.0003	-0.0001	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
162.00	-0.0010	-0.0013	-0.0012	-0.0008	-0.0005	-0.0003	-0.0001	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
166.00	-0.0010	-0.0012	-0.0011	-0.0008	-0.0005	-0.0003	-0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
169.00	-0.0009	-0.0012	-0.0011	-0.0007	-0.0005	-0.0003	-0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
172.00	-0.0009	-0.0011	-0.0010	-0.0007	-0.0004	-0.0003	-0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
174.00	-0.0008	-0.0010	-0.0010	-0.0006	-0.0004	-0.0003	-0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
176.00	-0.0007	-0.0009	-0.0009	-0.0006	-0.0004	-0.0002	-0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
177.00	-0.0007	-0.0008	-0.0008	-0.0005	-0.0003	-0.0002	-0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
178.00	-0.0006	-0.0007	-0.0007	-0.0005	-0.0003	-0.0002	-0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	-0.
179.00	-0.0004	-0.0006	-0.0005	-0.0003	-0.0002	-0.0001	-0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	-0.
180.00	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.

01 EXIT IN TEST

APPENDIX B

LISTING OF FORTRAN PROGRAM

The following computer program should be considered a subroutine, or black box for which an executive program is written. Such a program and sample test run are given in appendix A.

Subroutine POISS1 generates the r-θ mesh then generates and inverts the requisite coefficient matrices. POISS2 then uses these matrices together with the right-hand side values to generate the solution.

```
$IBFTC POIS1

       SUBROUTINE POISS1
C
       COMMON/BLOCK1/NR,NT,RA(17),TA(25),X(25,16),X1,AVS
       COMMON/BLOCK2/V(25,25,15),A(25,15),FM(25,15),BCMC(25),BCMR(16),
      1 BCMR1(25),BCMA(16),V1,FM1,NRM1,NRM2,NRM3,NTM1,RBA(16),DRA(16),
      2 NTE,NTEM1,TBA(25),DTA(25),STBA(25),ROW(25),D(25,25)
       LOGICAL AVS
C
       NRM1=NR-1
       NRM2=NR-2
       NRM3=NR-3
       NTM1=NT-1
C
C         CHECK MESH SIZE.
       IF(NR.GT.17) CALL ARERR(23HTOO MANY R MESH LINES $)
       IF(NT.GT.25) CALL ARERR(27HTOO MANY THETA MESH LINES $)
C
C         GENERATE RBAR AND DELR ARRAYS.
       DO 1 IR=1,NRM1
       RBA(IR)= .5*(RA(IR)+RA(IR+1))
     1 DRA(IR)= RA(IR+1)-RA(IR)
C
C         GENERATE THETA-BAR, SIN THETA-BAR
C         AND DEL-THETA ARRAYS.
       DO 2 IT=1,NTM1
       TBA(IT)= .5*(TA(IT)+TA(IT+1))
       STBA(IT)= SIN(TBA(IT))
     2 DTA(IT)= TA(IT+1)-TA(IT)
C
C         TEST CONFIGURATION FOR FIRST PASS.
       IF(TA(1).EQ.0.)GO TO 3
       NTE=NT
       IF(AVS) NTE=NT-1
       NTEM1=NTE-1
       CALL SEG1
       RETURN
C
C         FULL SPHERE SOLUTION (1ST PASS).
     3 CALL SPHER1
       RETURN
C
       ENTRY POISS2
       IF(TA(1).EQ.0.) GO TO 4
       CALL SEG2
       RETURN
C
C         FULL SPHERE SOLUTION (2ND PASS).
     4 CALL SPHER2
C
       RETURN
       END
```

```
$IBFTC SPHR1
      SUBROUTINE SPHER1
C        GENERATES AND INVERTS COEFFICIENT MATRICES
C        USED FOR FULL SPHERE SOLUTION.
C             NR-2    TRIDIAGONAL BLOCK MATRICES.
C          2(NR-2)-1    DIAGONAL BLOCK MATRICES.
C
      COMMON/BLOCK1/NR,NT,RA(17),TA(25),X(25,16),X1,AVS
      COMMON/BLOCK2/V(25,25,15),A(25,15),FM(25,15),BCMC(25),BCMR(16),
     1 BCMR1(25),BCMA(16),V1,FM1,NRM1,NRM2,NRM3,NTM1,RBA(16),DRA(16),
     2 NTE,NTEM1,TBA(25),DTA(25),STBA(25),ROW(25),D(25,25)
      LOGICAL AVS
C
C        INVERSE OF CENTER POINT.
      V1= 1./AE(1,1,0)
      FM1= FE(1,1)
      K1=1
C
      DO 14 IR=2,NRM1
      GO TO (1,3),K1
C
C        IR=2 ONLY.
    1 DO 2 IT1=1,NT
      A(IT1,1)= AE(2,IT1,1)
      FM(IT1,1)= FE(2,IT1)
    2 BCMC(IT1)=-AE(NRM1,IT1,2)
      GO TO 5
C
    3 DO 4 IT1=1,NT
      A(IT1,IR-1)= AE(IR,IT1,1)
    4 FM(IT1,IR-1)= FE(IR,IT1)
C
C        GENERATE NR-2 NTXNT MATRICES.
C        (ONE FOR EACH R=CONSTANT LINE).
    5 DO 12 IT1=1,NT
      GO TO(6,8),K1
C
C        FORM -A2*V1*A2.
    6 DO 7 IT2=1,NT
    7 D(IT1,IT2)=-A(IT1,1)*V1*A(IT2,1)
      GO TO 10
C
C        FORM -A(I)*V(I-1)*A(I).
    8 DO 9 IT2=1,NT
    9 D(IT1,IT2)=-A(IT1,IR-1)*V(IT1,IT2,IR-2)*A(IT2,IR-1)
   10 IF(IT1.EQ.1) GO TO 11
C
C        ADD TRIDIAGONAL ELEMENTS OF
C        B(I) TO -A(I)*V(I-1)*A(I).
      D(IT1,IT1-1)= AE(IR,IT1,3)+D(IT1,IT1-1)
   11 D(IT1,IT1)= AE(IR,IT1,0)+D(IT1,IT1)
      IF(IT1.EQ.NT) GO TO 12
      D(IT1,IT1+1)= AE(IR,IT1,4)+D(IT1,IT1+1)
   12 CONTINUE
C
      CALL FACTOR(D,ROW,NT)
      CALL INVERT(D,ROW,NT,V(1,1,IR-1))
      GO TO (13,14),K1
   13 K1=2
   14 CONTINUE
C
      RETURN
      END
```

```
$IBFTC SPHR2
      SUBROUTINE SPHER2
C
      COMMON/BLOCK1/NR,NT,RA(17),TA(25),X(25,16),X1,AVS
      COMMON/BLOCK2/V(25,25,15),A(25,15),FM(25,15),BCMC(25),BCMR(16),
     1 BCMR1(25),BCMA(16),V1,FM1,NRM1,NRM2,NRM3,NTM1,RBA(16),DRA(16),
     2 NTE,NTEM1,TBA(25),DTA(25),STBA(25),ROW(25),D(25,25)
      LOGICAL AVS
C
      DOUBLE PRECISION T,Y(25)
C         CALCULATE G1.
      X1= V1*FM1*X1
C
C         CALCULATE G2 (IR=2).
      DO 1 IT=1,NT
    1 Y(IT)= FM(IT,1)*X(IT,1)-A(IT,1)*X1
      DO 3 IT1=1,NT
      T=0.D0
      DO 2 IT2=1,NT
    2 T=T+V(IT1,IT2,1)*Y(IT2)
    3 X(IT1,1)=T
C
C         CALCULATE G3 TO G(NRM2).
      DO 6 IR=3,NRM2
      DO 4 IT=1,NT
    4 Y(IT)= FM(IT,IR-1)*X(IT,IR-1)-A(IT,IR-1)*X(IT,IR-2)
      DO 6 IT1=1,NT
      T=0.D0
      DO 5 IT2=1,NT
    5 T=T+V(IT1,IT2,IR-1)*Y(IT2)
    6 X(IT1,IR-1)=T
C
C         CALCULATE G(NRM1).
      DO 7 IT=1,NT
    7 Y(IT)=FM(IT,NRM2)*X(IT,NRM2)+BCMC(IT)*X(IT,NRM1)-A(IT,NRM2)*
     1 X(IT,NRM3)
      DO 9 IT1=1,NT
      T= 0.D0
      DO 8 IT2=1,NT
    8 T=T+V(IT1,IT2,NRM2)*Y(IT2)
    9 X(IT1,NRM2)=T
C
C         BACKWARD SUBSTITUTION FOR SOLUTION.
      DO 11 IRC=1,NRM3
      IR= NRM1-IRC
      DO 11 IT1=1,NT
      T= 0.D0
      DO 10 IT2=1,NT
   10 T=T+V(IT1,IT2,IR-1)*A(IT2,IR)*X(IT2,IR)
   11 X(IT1,IR-1)= X(IT1,IR-1)-T
C
C         CALCULATE CENTER POINT.
      T= 0.D0
      DO 12 IT1=1,NT
   12 T=T+A(IT1,1)*X(IT1,1)
      X1= X1-V1*T
C
      RETURN
      END
```

```
$IBFTC SEGM1
      SUBROUTINE SEG1
C
C              BOUNDARY VALUE COEFFICIENTS.
C        BCMA -AXIAL VALUES (THETA=PI).
C        BCMC -SURFACE VALUES (CIRCUMFERENCE).
C        BCMR -RADIAL LINE VALUES (THETA=CONST. SURFACE).
C        BCMR1-CENTRAL VALUES.
C
C
      COMMON/BLOCK1/NR,NT,RA(17),TA(25),X(25,16),X1,AVS
      COMMON/BLOCK2/V(25,25,15),A(25,15),FM(25,15),BCMC(25),BCMR(16),
     1 BCMR1(25),BCMA(16),V1,FM1,NRM1,NRM2,NRM3,NTM1,RBA(16),DRA(16),
     2 NTE,NTEM1,TBA(25),DTA(25),STBA(25),ROW(25),D(25,25)
      LOGICAL AVS
C
      DO 1 IR=2,NRM1
      BCMR(IR)=-AE(IR,2,3)
    1 IF(AVS) BCMA(IR)=-AE(IR,NTE,4)
C
      DO 3 IT1=1,NTE
C        INITIALIZE.
      DO 2 IT2=1,NTE
    2 D(IT1,IT2)=0.0
      BCMC(IT1)=-AE(NRM1,IT1,2)
      BCMR1(IT1)=-AE(2,IT1,1)
      FM(IT1,1)=FE(2,IT1)
      IF(IT1.EQ.1) GO TO 3
C
C        GENERATE ELEMENTS OF B1 THEN INVERT TO GET V1.
C         (T,D,B COEFFS. FOR IR=2 MESH LINE).
      IF(IT1.NE.2) D(IT1-1,IT1-2)=AE(2,IT1,3)
      D(IT1-1,IT1-1)=AE(2,IT1,0)
      IF(IT1.NE.NTE) D(IT1-1,IT1)=AE(2,IT1,4)
    3 CONTINUE
      CALL FACTOR(D,ROW,NTEM1)
      CALL INVERT(D,ROW,NTEM1,V(1,1,1))
C
      DO 7 IR=3,NRM1
      DO 4 IT1=2,NTE
      A(IT1,IR-1)=AE(IR,IT1,1)
    4 FM(IT1,IR-1)=FE(IR,IT1)
C
C        GENERATE REMAINING VI MATRICES.
      DO 6 IT1=2,NTE
      DO 5 IT2=2,NTE
    5 D(IT1-1,IT2-1)=-A(IT1,IR-1)*V(IT1-1,IT2-1,IR-2)*A(IT2,IR-1)
      IF(IT1.NE.2) D(IT1-1,IT1-2)=D(IT1-1,IT1-2)+AE(IR,IT1,3)
      D(IT1-1,IT1-1)=D(IT1-1,IT1-1)+AE(IR,IT1,0)
    6 IF(IT1.NE.NTE) D(IT1-1,IT1)=D(IT1-1,IT1)+AE(IR,IT1,4)
      CALL FACTOR(D,ROW,NTEM1)
    7 CALL INVERT(D,ROW,NTEM1,V(1,1,IR-1))
C
      RETURN
      END
```

```
$IBFTC SEGM2
      SUBROUTINE SEG2
C
      COMMON/BLOCK1/NR,NT,RA(17),TA(25),X(25,16),X1,AVS
      COMMON/BLOCK2/V(25,25,15),A(25,15),FM(25,15),BCMC(25),BCMR(16),
     1 BCMR1(25),BCMA(16),V1,FM1,NRM1,NRM2,NRM3,NTM1,RBA(16),DRA(16),
     2 NTE,NTEM1,TBA(25),DTA(25),STBA(25),ROW(25),D(25,25)
      LOGICAL AVS
C
      DOUBLE PRECISION T,Y(25)
C
C           CALCULATE G1 (INNERMOST RING).
      DO 1 IT1=2,NTE
    1 Y(IT1)=FM(IT1,1)*X(IT1,1)+BCMR1(IT1)*X1
      Y(2)=Y(2)+BCMR(2)*X(1,1)
      IF(AVS) Y(NTE)=Y(NTE)+BCMA(2)*X(NT,1)
      DO 3 IT1=2,NTE
      T=0.D0
      DO 2 IT2=2,NTE
    2 T=T+V(IT1-1,IT2-1,1)*Y(IT2)
    3 X(IT1,1)=T
C
C           CALCULATE G(IR-1) (INTERIOR
C           RINGS 2 TO NRM3).
      DO 6 IR=3,NRM2
      DO 4 IT1=2,NTE
    4 Y(IT1)=FM(IT1,IR-1)*X(IT1,IR-1)-A(IT1,IR-1)*X(IT1,IR-2)
      Y(2)=Y(2)+BCMR(IR   )*X(1,IR-1)
      IF(AVS) Y(NTE)=Y(NTE)+BCMA(IR)*X(NT,IR-1)
      DO 6 IT1=2,NTE
      T=0.D0
      DO 5 IT2=2,NTE
    5 T=T+V(IT1-1,IT2-1,IR-1)*Y(IT2)
    6 X(IT1,IR-1)=T
C
C           CALCULATE G(NR-2).
      DO 7 IT1=2,NTE
    7 Y(IT1)=FM(IT1,NRM2)*X(IT1,NRM2)+BCMC(IT1)*X(IT1,NRM1)
     1 -A(IT1,NRM2)*X(IT1,NRM3)
      Y(2)=Y(2)+BCMR(NRM1)*X( 1,NRM2)
      IF(AVS) Y(NTE)=Y(NTE)+BCMA(NRM1)*X(NT,NRM2)
      DO 9 IT1=2,NTE
      T=0.D0
      DO 8 IT2=2,NTE
    8 T=T+V(IT1-1,IT2-1,NRM2)*Y(IT2)
    9 X(IT1,NRM2)=T
C
C           BACKWARD SUBSTITUTION FOR SOLUTION.
      DO 11 IRC=1,NRM3
      IR=NRM1-IRC
      DO 11 IT1=2,NTE
      T=0.D0
      DO 10 IT2=2,NTE
   10 T=T+V(IT1-1,IT2 -1,IR-1)*A(IT2,IR)*X(IT2,IR)
   11 X(IT1,IR-1)=X(IT1,IR-1)-T
C
      RETURN
      END
```

```
$IBFTC AE1
      FUNCTION AE(I,J,K)
C
C        CALCULATION OF LEFT-HAND SIDE.
      COMMON/BLOCK1/NR,NT,RA(17),TA(25),X(25,16),X1,AVS
      COMMON/BLOCK2/V(25,25,15),A(25,15),FM(25,15),BCMC(25),BCMR(16),
     1 BCMR1(25),BCMA(16),V1,FM1,NRM1,NRM2,NRM3,NTM1,RBA(16),DRA(16),
     2 NTE,NTEM1,TBA(25),DTA(25),STBA(25),ROW(25),D(25,25)
      LOGICAL AVS
C
      M=NT
      IF(I.EQ.1) GO TO 17
      IF(J.EQ.1) GO TO 6
      IF(J.EQ.M) GO TO 11
C
C        REGION 1 (INTERIOR POINTS).
      IF(K.EQ.0) GO TO 5
      GO TO (1,2,3,4),K
C
C        L-COEFFICIENT.
    1 AE=(RBA(I-1)*(TBA(J)-TBA(J-1)))*(RBA(I-1)*STBA(J)+RBA(I-1)
     1 *STBA(J-1))/(2.*DRA(I-1))
      RETURN
C
C        R-COEFFICIENT.
    2 AE=(RBA(I)*(TBA(J)-TBA(J-1)))*(RBA(I)*STBA(J)+RBA(I)*STBA(J-1))
     1 /(2.*DRA(I))
      RETURN
C
C        T-COEFFICIENT.
    3 AE=(RBA(I)-RBA(I-1))*(RBA(I)*STBA(J-1)+RBA(I-1)*STBA(J-1))
     1 /(2.*RA(I)*DTA(J-1))
      RETURN
C
C        B-COEFFICIENT.
    4 AE=(RBA(I)-RBA(I-1))*(RBA(I)*STBA(J)+RBA(I-1)*STBA(J))
     1 /(2.*RA(I)*DTA(J))
      RETURN
C
C        D-COEFFICIENT=L+R+T+B.
    5 AE=-((RBA(I-1)*(TBA(J)-TBA(J-1)))*(RBA(I-1)*STBA(J)+RBA(I-1)
     1 *STBA(J-1))/(2.*DRA(I-1))
     2  +(RBA(I)*(TBA(J)-TBA(J-1)))*(RBA(I)*STBA(J)+RBA(I)*STBA(J-1))
     3 /(2.*DRA(I))
     4  +(RBA(I)-RBA(I-1))*(RBA(I)*STBA(J-1)+RBA(I-1)*STBA(J-1))
     5 /(2.*RA(I)*DTA(J-1))
     6  +(RBA(I)-RBA(I-1))*(RBA(I)*STBA(J)+RBA(I-1)*STBA(J))
     7 /(2.*RA(I)*DTA(J)))
      RETURN
C
C        REGION 2 (THETA=THETAO BOUNDARY LINE, J=1).
```

```
    6 IF(K.EQ.0) GO TO 10
      GO TO (7,8,16,9),K
C
C         L-COEFFICIENT.
    7 AE=(RBA(I-1)*(TBA(1)-TA(1)))*(RBA(I-1)*STBA(1))/(2.*DRA(I-1))
      RETURN
C
C         R-COEFFICIENT.
    8 AE=(RBA(I)*(TBA(1)-TA(1)))*(RBA(I)*STBA(1))/(2.*DRA(I))
      RETURN
C
C         B-COEFFICIENT.
    9 AE=(RBA(I)-RBA(I-1))*(RBA(I)*STBA(1)+RBA(I-1)*STBA(1))
     1 /(2.*RA(I)*DTA(1))
      RETURN
C
C         D-COEFFICIENT.
   10 AE=-((RBA(I-1)*(TBA(1)-TA(1)))*(RBA(I-1)*STBA(1))/(2.*DRA(I-1))
     1  +(RBA(I)*(TBA(1)-TA(1)))*(RBA(I)*STBA(1))/(2.*DRA(I))
     2  +(RBA(I)-RBA(I-1))*(RBA(I)*STBA(1)+RBA(I-1)*STBA(1))
     3 /(2.*RA(I)*DTA(1)))
      RETURN
C
C         REGION 3 (NEGATIVE AXIAL BOUNDARY, THETA=PI).
   11 IF(K.EQ.0) GO TO 15
      GO TO (12,13,14,16),K
C
C         L-COEFFICIENT.
   12 AE=(RBA(I-1)*(TA(M)-TBA(M-1)))*(RBA(I-1)*STBA(M-1))/(2.*DRA(I-1))
      RETURN
C
C         R-COEFFICIENT.
   13 AE=(RBA(I)*(TA(M)-TBA(M-1)))*(RBA(I)*STBA(M-1))/(2.*DRA(I))
      RETURN
C
C         T-COEFFICIENT.
   14 AE=(RBA(I)-RBA(I-1))*(RBA(I)*STBA(M-1)+RBA(I-1)*STBA(M-1))
     1 /(2.*RA(I)*DTA(M-1))
      RETURN
C
C         D-COEFFICIENT.
   15 AE=((RBA(I-1)*(TA(M)-TBA(M-1)))*(RBA(I-1)*STBA(M-1))
     1 /(2.*DRA(I-1))
     2  +(RBA(I)*(TA(M)-TBA(M-1)))*(RBA(I)*STBA(M-1))/(2.*DRA(I))
     3  +(RBA(I)-RBA(I-1))*(RBA(I)*STBA(M-1)+RBA(I-1)*STBA(M-1))
     4 /(2.*RA(I)*DTA(M-1))) *(-1.)
      RETURN
C         (NO B-COEFFICIENT).
   16 AE=0.
      RETURN
C
C         REGION 4 (CENTER OF SPHERE, I=1).
   17 S= (RBA(1)*(TBA(1)-TA(1)))*(RBA(1)*STBA(1))/(2.*DRA(1))
      NM1=M-1
      DO 18 J1=2,NTM1
   18 S=S+(RBA(1)*(TBA(J1)-TBA(J1-1)))*(RBA(1)*STBA(J1)+RBA(1)
     1 *STBA(J1-1))/(2.*DRA(1))
      S=S+(RBA(1)*(TA(M)-TBA(M-1)))*(RBA(1)*STBA(M-1))/(2.*DRA(1))
      AE=-S
      RETURN
      END
```

```
$IBFTC FE1
      FUNCTION FE(I,J)
C        CALCULATION OF RIGHT-HAND SIDE.
C
      COMMON/BLOCK1/NR,NT,RA(17),TA(25),X(25,16),X1,AVS
      COMMON/BLOCK2/V(25,25,15),A(25,15),FM(25,15),BCMC(25),BCMR(16),
     1 BCMR1(25),BCMA(16),V1,FM1,NRM1,NRM2,NRM3,NTM1,RBA(16),DRA(16),
     2 NTE,NTEM1,TBA(25),DTA(25),STBA(25),ROW(25),D(25,25)
      LOGICAL AVS
C
      M=NT
      IF(I.EQ.1) GO TO 3
      IF(J.EQ.1) GO TO 1
      IF(J.EQ.NT) GO TO 2
C
C        REGION 1 (INTERIOR POINTS).
      FE=(RBA(I)-RBA(I-1))*(RBA(I)*(TBA(J)-TBA(J-1))+RBA(I-1)
     1 *(TBA(J)-TBA(J-1)))*(RBA(I)*(STBA(J)+STBA(J-1))+RBA(I-1)
     2 *(STBA(J)+STBA(J-1)))/8.
      RETURN
C
C        REGION 2 (THETA=THETAO BOUNDARY LINE, J=1).
    1 FE=(RBA(I)-RBA(I-1))*(RBA(I)*(TBA(1)-TA(1))+RBA(I-1)
     1 *(TBA(1)-TA(1)))*((RBA(I)+RBA(I-1))*STBA(1))/8.
      RETURN
C
C        REGION 3 (NEGATIVE AXIAL BOUNDARY, THETA=PI).
    2 FE=(RBA(I)-RBA(I-1))*(RBA(I)*(TA(M)-TBA(M-1))+RBA(I-1)
     1 *(TA(M)-TBA(M-1)))*((RBA(I)+RBA(I-1))*STBA(M-1))/8.
      RETURN
C
C        REGION 4 (CENTER OF SPHERE, I=1).
    3 S=(TBA(1)-TA(1))*STBA(1)
      MM1=M-1
      DO 4 J1=2,MM1
    4 S=S+(TBA(J1)-TBA(J1-1))*(STBA(J1)+STBA(J1-1))
      S=S+(TA(M)-TBA(M-1))*STBA(M-1)
      FE=RBA(1)**3 *S/6.
      RETURN
      END
```

```
$IBFTC FAC1
      SUBROUTINE FACTOR(A,ROW,ORDER)
C         SEE COMPUTATIONAL METHODS OF LINEAR ALGEBRA
C         BY FADEEVA (PAGE 86), REFERENCE 2.
C
      INTEGER ROW,ORDER
      DIMENSION A(25,25),ROW(25)
      DOUBLE PRECISION T,ZERO
      REAL MAX
C
      DATA ZERO/0.0D0/
      N=ORDER
      ASSIGN 1 TO K1
      ASSIGN 9 TO K2
      DO 14 I=1,N
      IP1=I+1
      IM1=I-1
      GO TO K1,(2,1)
    1 ASSIGN 2 TO K1
      GO TO 5
C
C         CALCULATION OF LOWER TRIANGULAR MATRIX.
    2 DO 4 L=I,N
      T=ZERO
      DO 3 K=1,IM1
    3 T=T+A(K,I)*A(L,K)
    4 A(L,I)=A(L,I)-T
C
C         DETERMINE MAXIMUM ELEMENT IN COLUMN I.
    5 MAX=0.0
      DO 6 L=I,N
      IF(ABS(A(L,I)).LE.ABS(MAX))GO TO 6
      NI=L
      MAX=A(L,I)
    6 CONTINUE
C
      IF(MAX.EQ.0.0)CALL ARERR(37H SUBR. FACTOR HAS A SINGULAR MATRIX-$)
      ROW(I)=NI
      IF(I.NE.NI)GO TO 16
      IF(I.EQ.N)RETURN
      GO TO 8
C
C         INTERCHANGE ROWS I AND NI.
   16 DO 7 K=1,N
      S=A(I,K)
      A(I,K)=A(NI,K)
    7 A(NI,K)=S
C
C         DIVIDE ELEMENTS OF ROW 1
C         BY MAXIMUM ELEMENT A(1,1).
    8 GO TO K2,(11,9)
    9 DO 10 L=2,N
   10 A(1,L)=A(1,L)/MAX
      ASSIGN 11 TO K2
      GO TO 14
C
C         CALCULATION OF UPPER TRIANGULAR MATRIX.
   11 DO 13 L=IP1,N
      T=ZERO
      DO 12 K=1,IM1
   12 T=T+A(I,K)*A(K,L)
   13 A(I,L)=(A(I,L)-T)/MAX
C
   14 CONTINUE
C
      RETURN
      END
```

```
$IBFTC INV1
      SUBROUTINE INVERT(A,ROW,ORDER,D)
C        SEE COMPUTATIONAL METHODS OF LINEAR ALGEBRA
C        BY FADEEVA (PAGE 87), REFERENCE 2.
C
      INTEGER ROW,ORDER
      DIMENSION A(25,25),ROW(25),D(25,25)
      DOUBLE PRECISION T,ZERO
C
      DATA ZERO/0.0D0/
      N=ORDER
      ASSIGN 7 TO K1
      I=N
    1 IP1=I+1
      IM1=I-1
C
C        USING LOWER TRIANGULAR MATRIX
C        CALCULATE ELEMENTS OF L-INVERSE.
      DO 4 M=1,I
      L=IP1-M
      T=ZERO
      GO TO K1,(2,7)
    7 ASSIGN 2 TO K1
      GO TO 4
    2 LP1=L+1
      DO 3 K=LP1,N
    3 T=T+A(K,L)*D(I,K)
    4 D(I,L)=(FLOAT(L/I)-T)/A(L,L)
      IF(I.EQ.1)GO TO 8
C
C        USING UPPER TRIANGULAR MATRIX
C        CALCULATE ELEMENTS OF U-INVERSE.
      DO 6 M=1,IM1
      L=I-M
      T=ZERO
      LP1=L+1
       DO 5 K=LP1,N
    5 T=T+A(L,K)*D(K,I)
    6 D(L,I)=-T
      I=IM1
      GO TO 1
C
C         INTERCHANGE COLUMNS I AND K.
    8 I=N-1
    9 K=ROW(I)
      DO 10 L=1,N
      S=D(L,I)
      D(L,I)=D(L,K)
   10 D(L,K)=S
C
      IF(I.EQ.1)RETURN
      I=I-1
     GO TO 9
     END
```

APPENDIX C

FORTRAN SYMBOLS

FORTRAN variable	Definition
A(25, 15)	storage for diagonal matrices
AVS	axial value switch (used in cases II and III as a logical quantity)
BCMA(16)	boundary condition multiplier array on negative axis ($\theta = \pi$)
BCMC(25)	boundary condition multiplier array on surface of sphere
BCMR(16)	boundary condition multiplier array on θ = constant surface of segment
BCMR1(25)	boundary condition multiplier at central point
D(25, 25)	matrix storage area
DRA(16)	Δr array
DTA(25)	$\Delta\theta$ array
FM(25, 15)	F (right-hand side) multiplier array
FM1	coefficient of center point
I	looping index
IM1	I - 1
IP1	I + 1
IR, IRC	r indexes
IT, IT1, IT2	θ indexes
J, J1 K, K1, K2	miscellaneous switching integers
L, LP1 M, MM1	looping indexes
MAX	maximum value storage
N	order of matrix
NI	matrix row index
NR	number of r mesh lines

FORTRAN Variable	Definition
NRM1, NRM2, NRM3	NR-1, NR-2, NR-3
NT	number of θ mesh lines for full sphere
NTE	number of θ mesh lines for spherical segment
NTEM1	NTE - 1
NTM1	NT - 1
ORDER	order of matrix
RA(17)	array of r mesh values
RBA(16)	$\bar{r}$ array
ROW(25)	matrix row indicators
S	summation storage
STBA(25)	sin $\bar{\theta}$
T	temporary storage
TA(25)	array of θ mesh values
TBA(25)	$\bar{\theta}$ array
V(25, 25, 15)	25 by 25 matrix storage area; stores up to 15 inverted matrices
V1	inverse of center point
X(25, 16)	interior mesh values
X1	central mesh value
Y(25)	double precision temporary storage array
ZERO	double precision initializer

APPENDIX D

FLOW CHARTS

In the following flow charts (figs. 12 to 21) all diamond shapes refer to decision making points in the subroutines. All other operations are indicated with rectangular blocks. Numbers appearing on the upper left of any block refer to the corresponding statement numbers in the program. All FORTRAN symbols used are defined in appendix C.

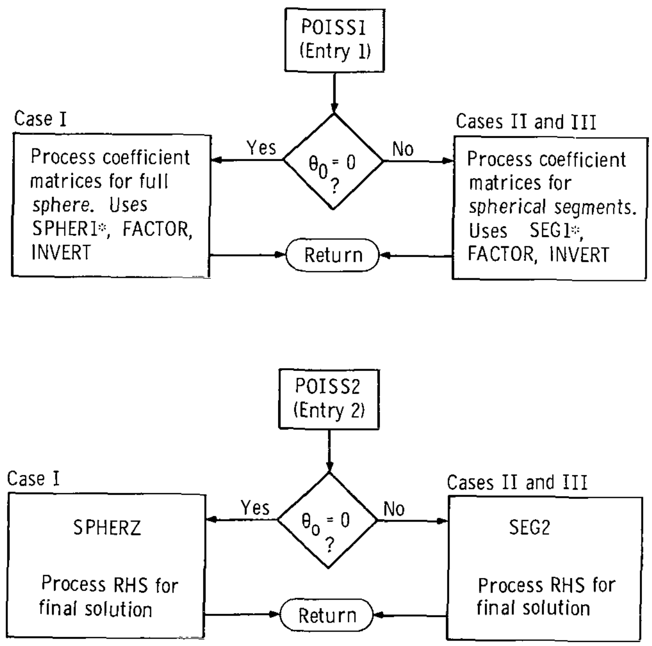


Figure 12. - Macro flow chart showing interrelationship of various subroutines used in program. (*Uses functions AE and FE.)

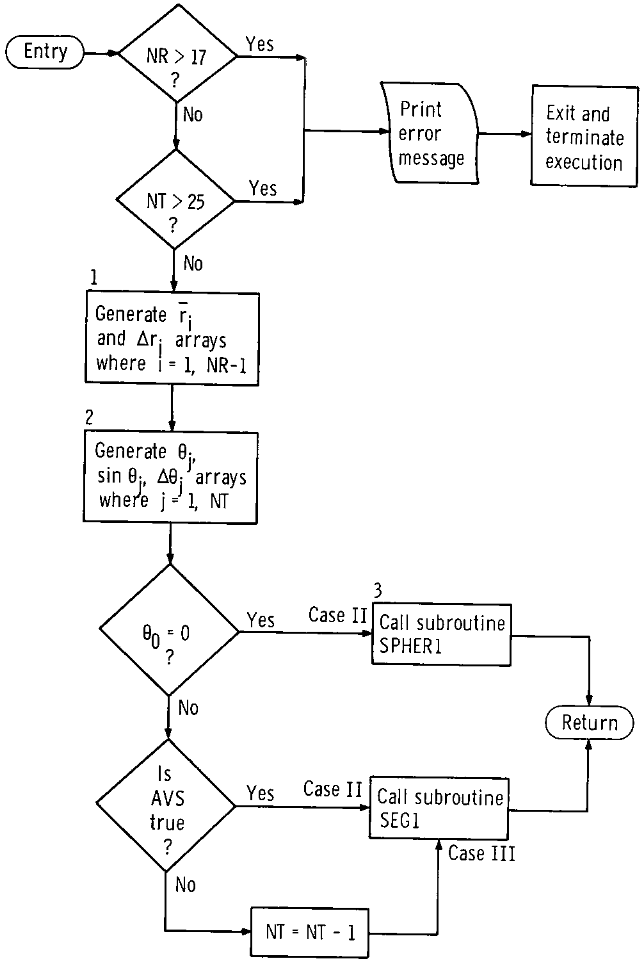


Figure 13. - Subroutine POISS1.

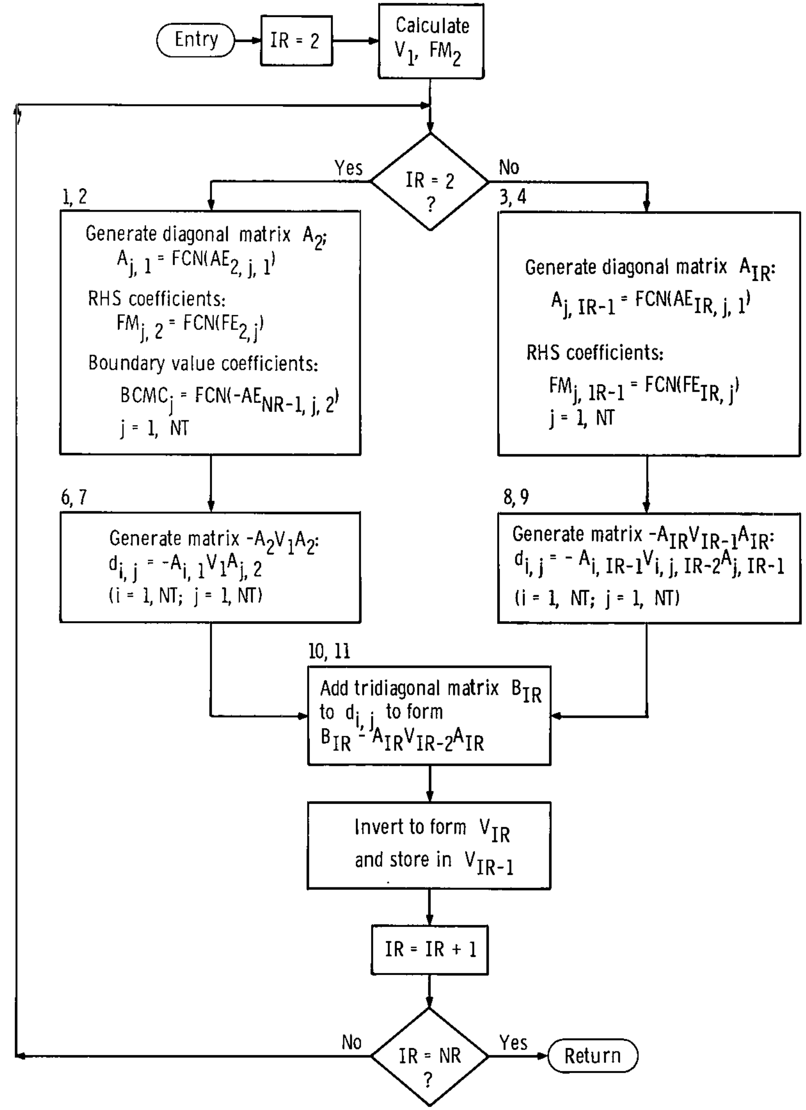


Figure 14. - Subroutine SPHER1.

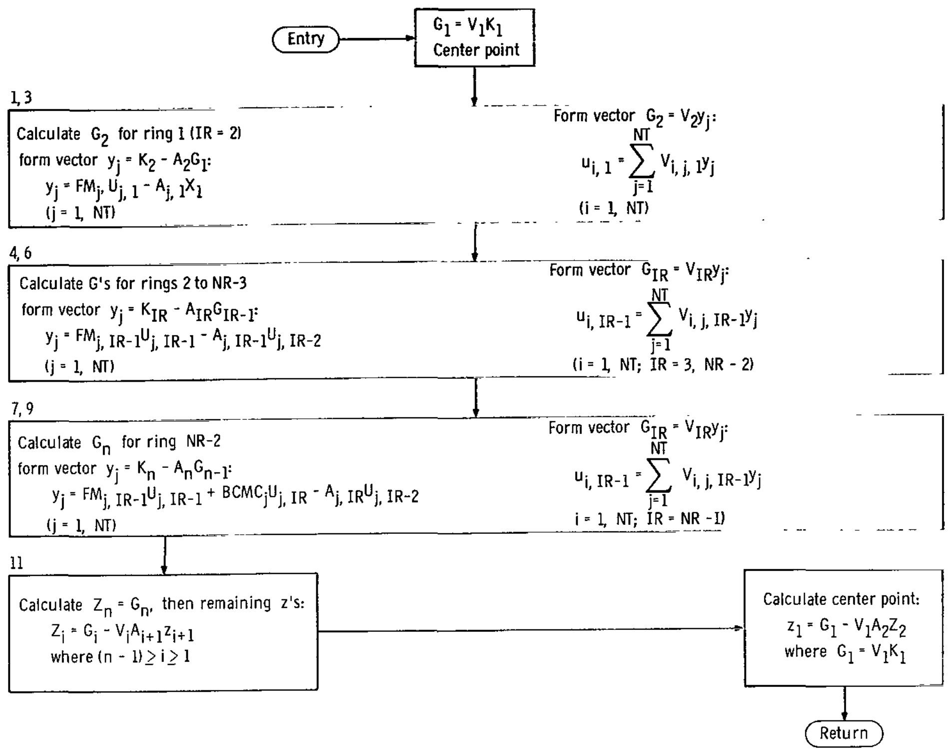


Figure 15. - Subroutine SPHER2.

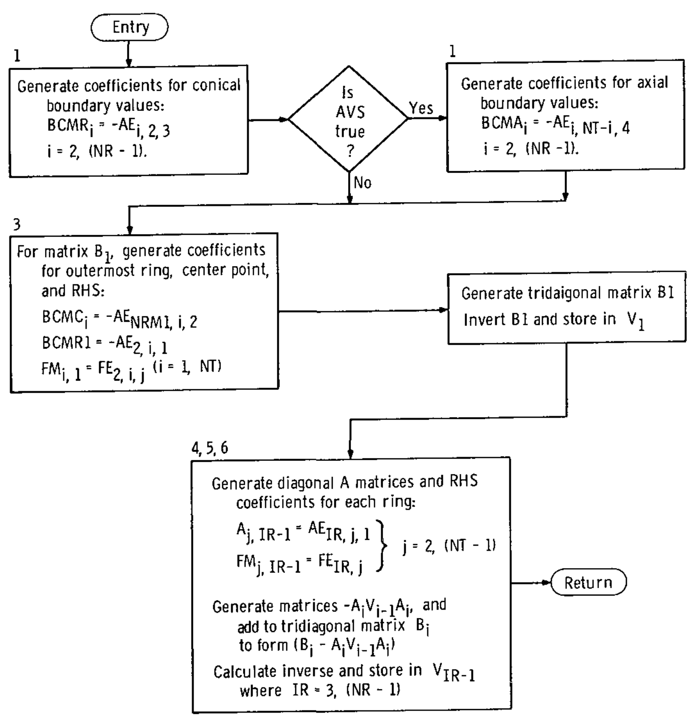


Figure 16. - Subroutine SEG1.

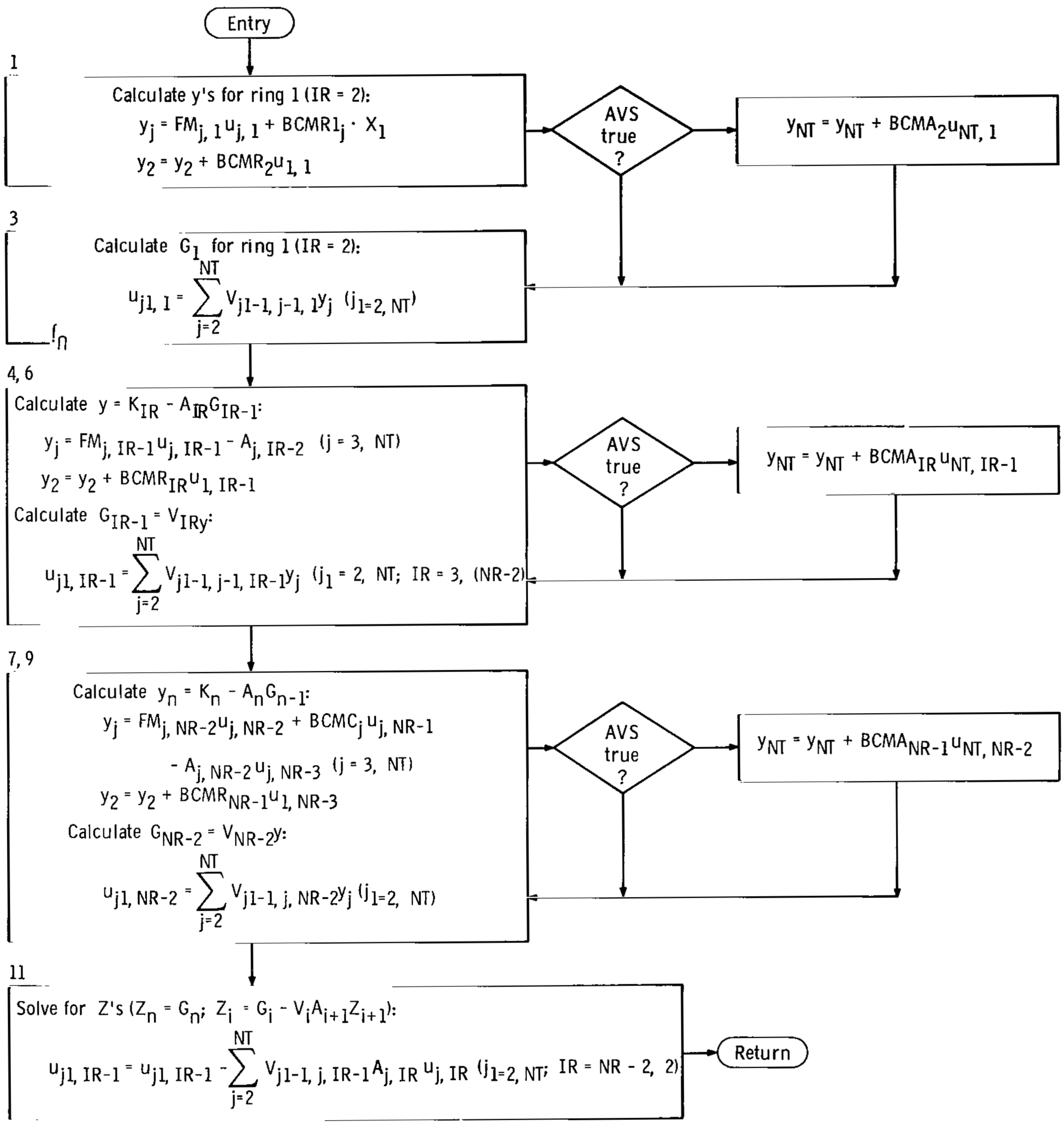


Figure 17. - Subroutine SEG 2.

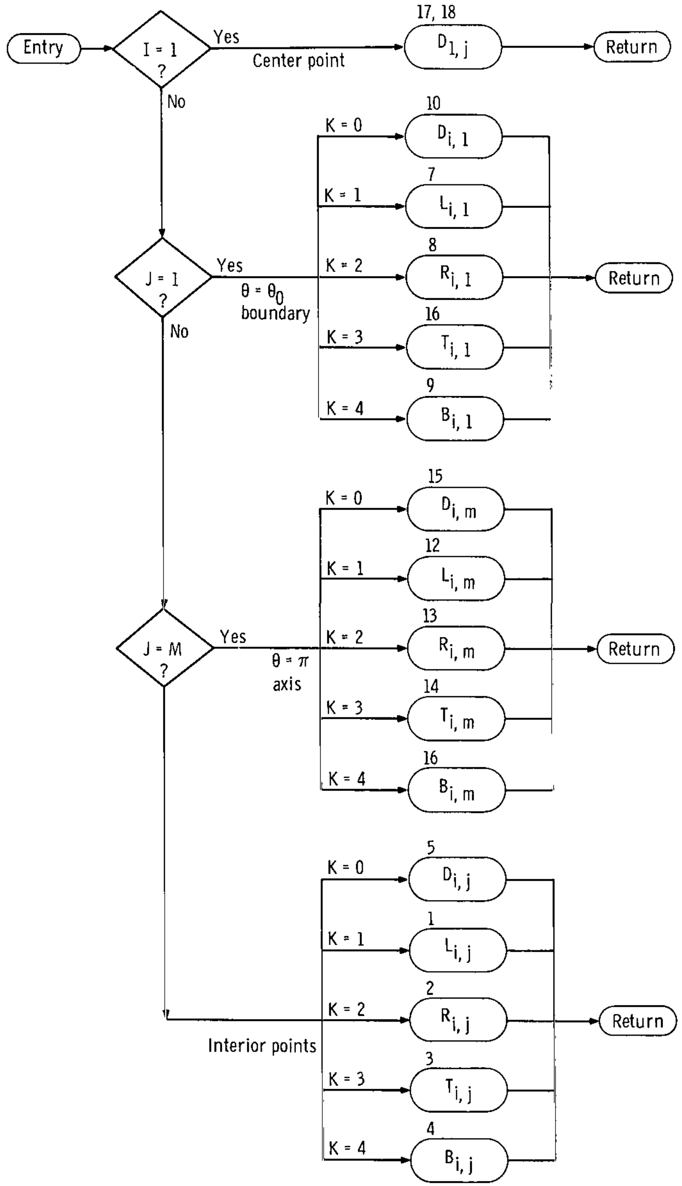


Figure 18. - Function AE(I, J, K) - calculation of LHS coefficients. (Formulas for quantities L, R, T, B, and D are given in the main text.)

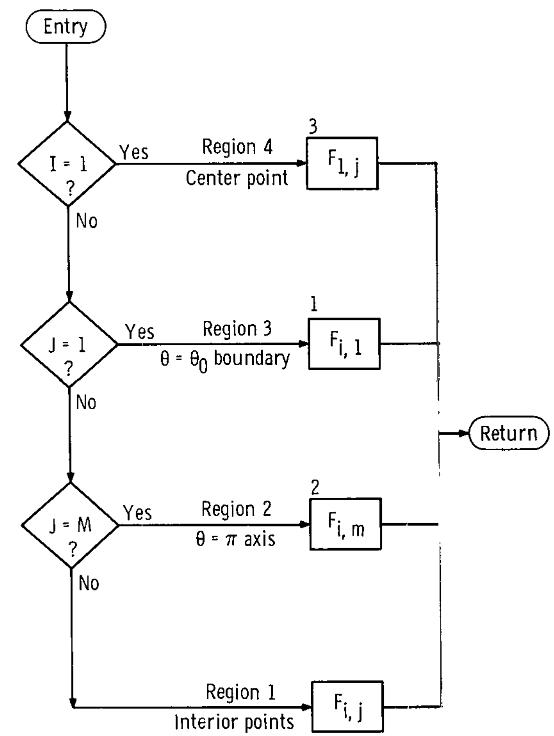


Figure 19. - Function FE(I, J) - calculation of RHS coefficients. (See text for formula for F.)

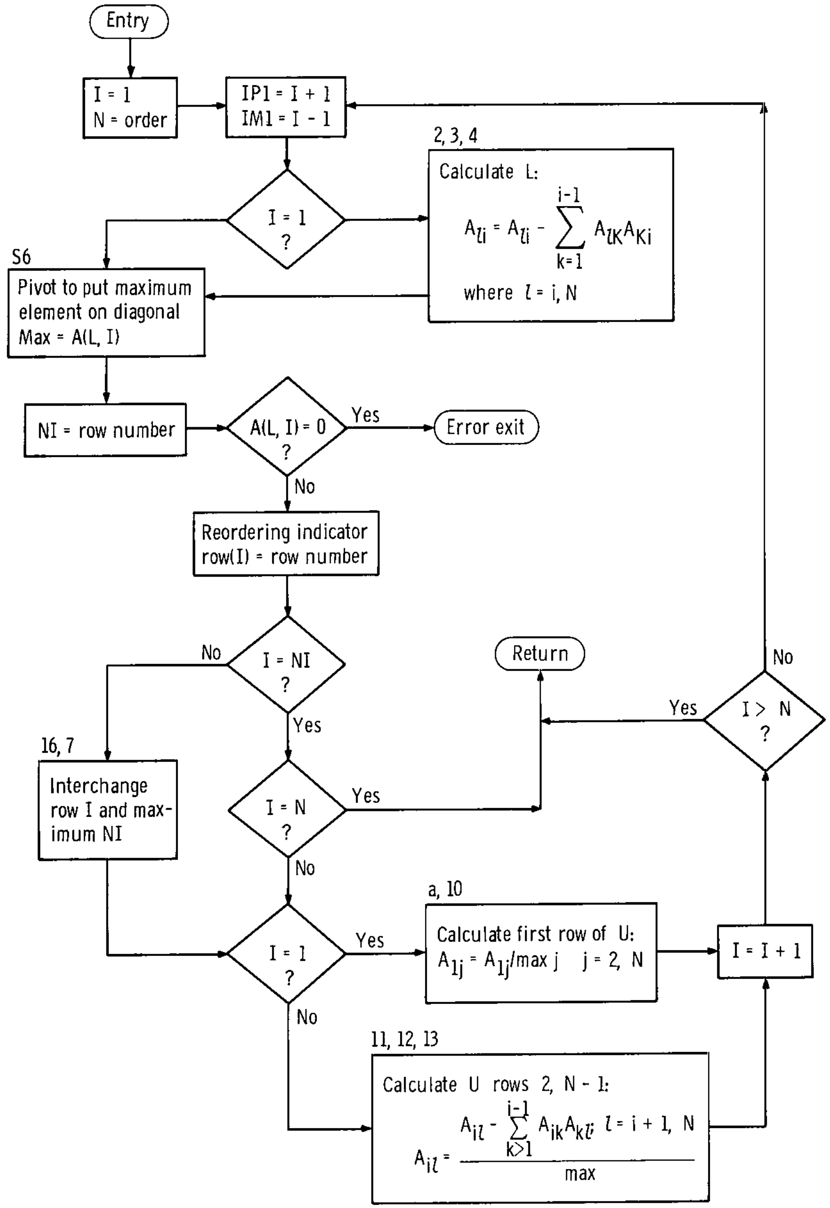


Figure 20. - Subroutine FACTOR.

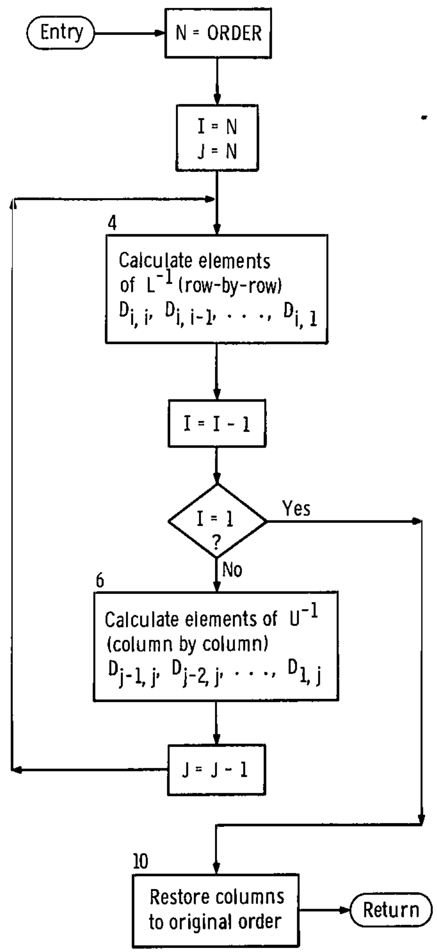


Figure 21. - Subroutine INVERT.

REFERENCES

1. Varga, Richard S.: Matrix Iterative Analysis. Prentice-Hall, Inc., 1962.

2. Faddeeva, V. N. (Curtis D. Benster, trans.): Computational Methods of Linear Algebra. Dover Publications, Inc., 1959.

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